

ALGEBRAIC FILLING INEQUALITIES AND COHOMOLOGICAL WIDTH

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ABSTRACT. In his work on singularities, expanders and topology of maps, Gromov showed, using isoparametric inequalities in graded algebras, that every real valued map on the n -torus admits a fibre whose homological size is bounded below by some universal constant depending on n . He obtained similar estimates for maps with values in finite dimensional complexes, by a Lusternik–Schnirelmann type argument.

We describe a new homological filling technique which enables us to derive sharp lower bounds in these theorems in certain situations. This partly realizes a programme envisaged by Gromov.

In contrast to previous approaches our methods imply similar lower bounds for maps defined on products of higher dimensional spheres.

1. INTRODUCTION

This paper is profoundly inspired by [8] and [9], in which, among others, the following two theorems were shown.

Theorem 1.1 ([9, pp. 424]). Let $k < \frac{n}{2}$ and let T^n denote the n -dimensional torus. Every continuous map $f: T^n \rightarrow \mathbb{R}$ admits a point $y \in \mathbb{R}$ such that the rank of the restriction homomorphism satisfies

$$\mathrm{rk} [H^k(T^n) \rightarrow H^k(f^{-1}(y))] \geq \left(1 - \frac{2k}{n}\right) \binom{n}{k}.$$

The second theorem is the so-called *maximal fiber inequality*:

Theorem 1.2 ([8, pp. 13], [9, Section 4.2]). Let Y^q be a q -dimensional simplicial complex and $n \geq p(q+1)$. Every continuous map $f: T^n \rightarrow Y$ admits a point $y \in Y$ satisfying

$$\mathrm{rk} [H^*T^n \rightarrow H^*(f^{-1}(y))] \geq 2^p.$$

In the theorems above and for the rest of the paper if X is a topological space H^*X shall denote Čech cohomology and the coefficient ring is \mathbb{Z} unless specified otherwise.

Definition 1.3 (Cohomological width). Let R be a coefficient ring such that the rank of a homomorphism between R -modules makes sense, e.g. \mathbb{Z} , \mathbb{Z}_2 or \mathbb{Q} . For every $y \in Y$ we can consider the rank of the Čech cohomology restriction homomorphism

$$H^*(X; R) \rightarrow H^*(f^{-1}y; R).$$

(i) For every continuous map $f: X \rightarrow Y$ the expressions

$$\begin{aligned}\text{width}_*(f; R) &:= \max_{y \in Y} \text{rk} [H^*(X; R) \rightarrow H^*(f^{-1}y; R)] \quad \text{and} \\ \text{width}_k(f; R) &:= \max_{y \in Y} \text{rk} [H^k(X; R) \rightarrow H^k(f^{-1}y; R)]\end{aligned}$$

are called the *total* or *degree k cohomological width* of f .

(ii) For fixed topological spaces X and Y the minima

$$\begin{aligned}\text{width}_*(X/Y; R) &:= \min_{f \in C(X, Y)} \text{width}_*(f; R) \quad \text{and} \\ \text{width}_k(X/Y; R) &:= \min_{f \in C(X, Y)} \text{width}_k(f; R)\end{aligned}$$

where $C(X, Y)$ denotes the set of all continuous maps $f: X \rightarrow Y$ are called the *total* or *degree k cohomological width* of X over Y .

For every continuous map $f: X \rightarrow Y$ the preimages of points are called the *fibers* of f and $\text{width}_k(f)$ gives a lower bound for the topological complexity of one fiber of f . The expression $\text{width}_k(X/Y)$ measures the complexity of X in terms of continuous maps to Y .

Up to now Theorems 1.1 and 1.2 have been essentially the only two inequalities about cohomological width. In this paper we will give new lower bounds for $\text{width}_k(X/Y)$ where X and Y are fixed manifolds.

A careful analysis of the proof of Theorem 1.2 shows that this $y \in Y$ actually satisfies

$$\text{rk} [H^k(T^n) \rightarrow H^k(f^{-1}(y))] \geq \binom{p}{k} \quad (1.1)$$

for every $0 \leq k \leq p$.

We can compare the different lower bounds, e.g. for $\text{width}_k(T^{2p}/\mathbb{R})$: Theorem 1.1 yields

$$\text{width}_k(T^{2p}/\mathbb{R}) \geq \left(1 - \frac{k}{p}\right) \binom{2p}{k} \quad (1.2)$$

whereas we get from Theorem 1.2 that

$$\text{width}_k(T^{2p}/\mathbb{R}) \geq \binom{p}{k}. \quad (1.3)$$

The bound (1.2) is significantly stronger than (1.3) but the latter holds for all 1-dimensional target spaces Y^1 , not just $Y^1 = \mathbb{R}$.

When investigating $\text{width}_k(X/Y)$ the dimension of the target space Y is called the *codimension* of the cohomological width problem. Theorem 1.1 is a codimension 1 result and its proof uses so-called *isoperimetric inequalities in algebras*. Theorem 1.2 on the other hand is a result admitting target spaces Y^q of arbitrary codimension $q \geq 1$. Its proof is far less geometric and uses Lusternik–Schnirelmann type argument. This argument and isoperimetric inequalities in algebras have been the only known techniques to prove cohomological waist inequalities.

Using a certain *filling argument in a space of $(n - q)$ -cycles in T^n* – which we will sketch in a moment – we could sharpen estimate (1.1) as follows.

Theorem 1.4. If N^q is a manifold we have

$$\text{width}_1(T^n/N) = n - q,$$

i.e. for every continuous $f: T^n \rightarrow N$ there exists a point $y \in N$ such that

$$\text{rk} [H^1(T^n) \rightarrow H^1(f^{-1}(y))] \geq n - q.$$

Any projection $f: T^n \rightarrow T^q$ shows that this inequality is sharp. A slightly more general construction shows equality can happen for every q -dimensional target manifold N .

It is the first non-trivial sharp evaluation of cohomological width, slightly improves the best known lower bound for $\text{width}_1(T^n/\mathbb{R})$ coming from Theorem 1.1 and generalises to arbitrary source manifolds that need not be tori but can be arbitrary essential m -manifolds with fundamental group \mathbb{Z}^n (cf. Theorem 3.24).

Gromov asked (cf. [9, Section 4.1 and Section 4.13 D]) whether one could use minimal models to prove cohomological waist inequalities. Using rational homotopy theory we could indeed prove the following estimate about cartesian powers of higher-dimensional spheres.

Theorem 1.5. Let $p \geq 3$ be odd and $n \leq p - 2$. Consider $M = (S^p)^n$ or any simply connected, closed manifold of dimension pn with the rational homotopy type $(S^p)_{\mathbb{Q}}^n$. For any orientable manifold N^q we have

$$\text{width}_p(M/N; \mathbb{Q}) \geq n - q.$$

We have not been able to remove the assumption $n \leq p - 2$ in this theorem but suspect that this can be done. Theorems 1.1 and 1.2 can be adapted to show $\text{width}_p((S^p)^n/\mathbb{R}) \geq n - 2$ and $\text{width}_p((S^p)^n/Y^q) \geq \frac{n}{q+1}$ but our bound is stronger.

The proofs of Theorems 1.4 and 1.5, which admit target manifolds of arbitrary codimension $q \geq 1$, use a new technique that is inspired by the metric filling argument sketched below. Theorem 1.5 is the first lower bound on width_p with $p > 1$ that has been proven using this technique.

Recall the important and classic *waist of the sphere inequality*:

Theorem 1.6. Every (for simplicity smooth and generic) \mathbb{R}^q -valued map f on the unit n -sphere admits a point $y \in \mathbb{R}^q$ such that the $(n - q)$ -dimensional Hausdorff volume satisfies

$$\text{vol}_{n-q} f^{-1}(y) \geq \text{vol}_{n-q} S^{n-q}$$

where $S^{n-q} \subset S^n$ is the $(n - q)$ -dimension equator in S^n .

Equality can happen e.g. if f is the restriction of a linear projection $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^q$.

Proof scheme (cf. [7, p. 134], [10]). We proceed by contradiction and assume that there is a smooth generic map $f: S^n \rightarrow \mathbb{R}^q$ such that every fiber $f^{-1}(y)$ has arbitrarily small $(n - q)$ -volume. Choose a generic and fine triangulation \mathcal{T} of \mathbb{R}^q . Fine means that the preimage of every k -simplex of \mathcal{T} has arbitrarily small $(n - q + k)$ -volume. For the preimages of the vertices of \mathcal{T} this holds by assumption, for the preimages of higher-dimensional simplices this can be achieved by subdivision. The sum of the preimages of the q -simplices represent the fundamental class $[S^n] \in H_n(S^n; \mathbb{Z})$. But using certain metric filling inequalities for $(n - q + l)$ -chains in S^n we can inductively construct a cone of $[S^n]$. This contradicts the

non-vanishing of the fundamental class $[S^n]$. Thus the assumption that all fibers $f^{-1}(y)$ can have arbitrarily small $(n - q)$ -volume must have been false. \square

We prove cohomological width inequalities by feeding this proof scheme with a new *cohomological filling inequality* (cf. Filling Lemma 3.13). This executes a plan that was indicated by Gromov in [9, Section 4.13 D].

This paper is organised as follows: in Section 2 we will recall the notion of ideal valued measures and their connection to cohomological width. The core ideas already appeared in [9, Section 4.1] but only as rough sketches and we allow ourselves the captatio benevolentiae and give rigorous statements and proofs. In particular we give a complete proof that the waist functional only increases under uniform limits, which allows us to reduce waist inequalities to the case of generic maps. This is important for our and possible further treatment of the subject. In Section 3 we define the space $cl^{n-q}(M)$ of $(n - q)$ -cycles in a manifold M such that every continuous map $f: M \rightarrow N$ between manifolds induces a non-trivial element in the homology of $cl^{n-q}(M)$. We show cohomological filling inequalities and use all of these ingredients to prove our waist inequalities.

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2. COHOMOLOGICAL WIDTH AND IDEAL VALUED MEASURES

We will use two cohomology theories, namely Čech cohomology and simplicial cohomology and it should always be clear from the context which one we mean depending on whether we evaluate it on topological spaces or simplicial complexes. Nevertheless and in order to avoid confusion we will consistently try to denote Čech cohomology by \check{H}^* and simplicial cohomology simply by H^* .

The following important observation motivates the rest of this section.

For every continuous map $f: X \rightarrow Y$ and every $y \in Y$ we have

$$\mathrm{rk} [\check{H}^* X \rightarrow \check{H}^* f^{-1}y] = \mathrm{rk} \left[\check{H}^* X / \ker [\check{H}^* X \rightarrow \check{H}^* f^{-1}y] \right].$$

Therefore we want to systematically study kernels of restriction homomorphisms $\check{H}^* X \rightarrow \check{H}^* C$ for closed subsets $C \subseteq X$ and this motivates the following

Definition 2.1 (Standard ideal valued measure, pushforward). Let X be a topological space and let $A := \check{H}^*(X)$. Let τ_X denote the system of all open subsets of X and $\mathcal{I}(A)$ the set of all graded ideals $I \subseteq A$.

- (i) Assign to every open $U \subseteq X$ a graded ideal $\mu_X(U) \subseteq A$ via

$$\begin{aligned} \mu_X: \tau_X &\rightarrow \mathcal{I}(A) \\ U &\mapsto \ker [\check{H}^* X \rightarrow \check{H}^*(X \setminus U)]. \end{aligned}$$

This map μ_X is called *the standard¹ ideal valued measure on X* (or $\mathcal{I}(A)$ -valued measure if one wants to emphasise the ambient algebra). It trivially satisfies $\mu_X(\emptyset) = 0$

¹Later on we will give an abstract definition of an ideal valued measure and if X is a compact manifold μ_X will be an instance of it.

(normalisation), $\mu_X(U_1) \subseteq \mu_X(U_2)$ whenever $U_1 \subseteq U_2$ (monotonicity) and $\mu_X(X) = A$ (fullness).

(ii) For any continuous map $f: X \rightarrow Y$ the assignment

$$f_*\mu_X: \tau_Y \rightarrow \mathcal{I}(A)$$

$$U \mapsto \mu_X(f^{-1}U) = \ker [\check{H}^*X \rightarrow \check{H}^*(X \setminus f^{-1}U)]$$

is called the *pushforward of μ_X along f* . It also satisfies normalisation, monotonicity and fullness. Note that $f_*\mu_X$ maps open subsets of Y to ideals in $\check{H}^*(X)$.

Corollary 2.2. For every continuous map $f: X \rightarrow Y$ and every closed subset $C \subseteq Y$ we have

$$\begin{aligned} \operatorname{rk} [\check{H}^*X \rightarrow \check{H}^*f^{-1}C] &= \operatorname{rk} \left[\check{H}^*X \Big/ \ker [\check{H}^*X \rightarrow \check{H}^*f^{-1}C] \right] \\ &= \operatorname{rk} \left[\check{H}^*X \Big/ f_*\mu_X(Y \setminus C) \right] \text{ and similarly} \\ \operatorname{rk} [H^kX \rightarrow H^kf^{-1}C] &= \operatorname{rk} \left[H^kX \Big/ f_*\mu_X(Y \setminus C) \cap H^kX \right]. \end{aligned}$$

Using specific features of Čech cohomology we want to derive more properties of μ_X and $f_*\mu_X$.

Proposition 2.3 (Additivity). Let X be a manifold and $A := \check{H}^*X$ its Čech cohomology algebra. The standard $\mathcal{I}(A)$ -valued measure μ_X on X satisfies *additivity*, i.e. for any two disjoint, open $U_1, U_2 \subseteq X$ we have

$$\mu_X(U_1 \dot{\cup} U_2) = \mu(U_1) + \mu(U_2).$$

The analogous statement also holds for the pushforward measure $f_*\mu_X$ along any continuous map $f: X \rightarrow Y$.

Repetition 2.4 (Čech cohomology, [4]). Let X be a topological space. For any open cover $\alpha = (U_i)_{i \in I}$ of X the *nerve* of α is a simplicial complex X_α with one vertex for every index $i \in I$ satisfying $U_i \neq \emptyset$ and an n -simplex $\{i_0, \dots, i_n\} \subset I$ belongs to X_α iff $U_{i_0} \cap \dots \cap U_{i_n} \neq \emptyset$. If α and β are two open coverings we write $\alpha < \beta$ if β refines α and this relation turns the set of open covers into a directed set. One can show that whenever $\alpha < \beta$ there is a well-defined map between simplicial cohomology groups $H^qX_\alpha \rightarrow H^qX_\beta$. The *Čech cohomology groups* are defined as the direct limit

$$\check{H}^q(X) := \varinjlim_{\alpha} H^q(X_\alpha).$$

Every element in this direct limit can be represented by a cohomology class $z \in H^q(X_\alpha)$ for a sufficiently fine open cover α of X . Čech cohomology satisfies the Eilenberg–Steenrod axioms (cf. [3]) and from this we get the following

Theorem 2.5 (Comparison between Čech and singular cohomology). For every CW pair (X, A) there is an isomorphism

$$\eta_{(X,A)}: \check{H}^*(X, A; R) \rightarrow H^*(X, A; R)$$

which defines a natural equivalence $\eta: \check{H}^* \rightarrow H^*$ of functors from CW pairs to R -modules.

The proof of Proposition 2.3 needs some preparation.

Lemma 2.6. Let X be a topological space, $V \subseteq X$ closed and $[z] \in \check{H}^q X$ such that $[z]|V = 0 \in \check{H}^q V$. Then there exists an open cover δ of X such that $[z]$ can be represented by some cohomology class $[z]_\delta \in H^q X_\delta$ and the restriction homomorphism satisfies

$$\begin{aligned} H^q X_\delta &\rightarrow H^q V_{\delta|V} \\ [z]_\delta &\mapsto 0 \end{aligned}$$

where $\delta|V$ denotes the induced open cover on V .

Proof. Every cohomology class $[z] \in \check{H}^q X = \varinjlim_\alpha H^q X_\alpha$ can be represented by some cohomology class $[z]_\alpha \in H^q(X_\alpha)$ for some sufficiently fine open cover α of X . Consider its image under the restriction homomorphism

$$\begin{aligned} H^q X_\alpha &\rightarrow H^q V_{\alpha|V} \\ [z]_\alpha &\mapsto [z]_\alpha|(\alpha|V). \end{aligned}$$

By assumption this restricted class vanishes in $\check{H}^q V = \varinjlim_\beta H^q V_\beta$, i.e. there exists a refinement γ of $\alpha|V$ such that the restriction satisfies

$$\begin{aligned} H^q V_{\alpha|V} &\rightarrow H^q V_\gamma \\ [z]_\alpha|(\alpha|V) &\mapsto 0. \end{aligned}$$

Since $V \subseteq X$ is closed we can extend γ to an open cover $\tilde{\gamma}$ of X . Let δ be a common refinement of α and $\tilde{\gamma}$, in particular $\delta|V$ refines γ .

On this level of the direct system we can represent $[z] \in \check{H}^q X$ by an element $[z]_\delta \in H^q X_\delta$ and its image under the restriction homomorphism of simplicial cohomology satisfies

$$H^q X_\delta \rightarrow H^q V_{\delta|V} \tag{2.1}$$

$$[z]_\delta \mapsto 0. \tag{2.2}$$

□

Lemma 2.7. Let (K, L) be a pair of simplicial complexes and $[z] \in H^q K$ a simplicial cohomology class such that $[z]|L = 0 \in H^q L$. Then $[z]$ can be represented by a simplicial q -cocycle $\tilde{z} \in C^q K$ such that $\tilde{z}|L = 0 \in C^q L$.

Proof. There exists a cochain $w \in C^{q-1} L$ such that $z|L = dw$. We can extend this cochain to a $(q-1)$ -cochain \tilde{w} on K by setting

$$\tilde{w}(\sigma) := \begin{cases} w(\sigma) & \text{if } \sigma \in L_{q-1} \\ 0 & \text{otherwise.} \end{cases}$$

For $\tilde{z} := z - d\tilde{w}$ and an arbitrary $\sigma \in L_q$ we have

$$\tilde{z}(\sigma) = (z - d\tilde{w})(\sigma) = z(\sigma) - d\tilde{w}(\sigma) = z(\sigma) - \tilde{w}(\partial\sigma) = z(\sigma) - w(\partial\sigma) = z(\sigma) - dw(\sigma) = 0. \quad \square$$

Lemma 2.8. Let K be a simplicial complex and L_1 and L_2 two subcomplexes of K such that $K = L_1 \cup L_2$. Let $[z] \in H^q K$ such that $[z]|L_1 \cap L_2 = 0 \in H^q(L_1 \cap L_2)$. Then there exists a decomposition $[z] = [z_1] + [z_2]$ for cohomology classes $[z_i] \in H^q K$ satisfying $z_i|L_i = 0 \in C^q L_i$ ($i = 1, 2$).

Proof. Using the preceding lemma we can assume that $z|_{L_1 \cap L_2} \in C^q(L_1 \cap L_2)$. For any $\sigma \in K_q$ define $z_1 \in C^q K$ by

$$z_1(\sigma) := \begin{cases} z(\sigma) & \text{if } \sigma \in (L_2)_q \\ 0 & \text{if } \sigma \in (L_1)_q. \end{cases}$$

and $z_2 \in C^q K$ analogously. Since $z|_{L_1 \cap L_2} = 0 \in C^q(L_1 \cap L_2)$ these cochains are well-defined, satisfy $z = z_1 + z_2$. They are cocycles since for every $\sigma \in (L_1)_q$ we have

$$dz_1(\sigma) = z_1(\partial\sigma) = 0$$

and for every $\sigma \in (L_2)_q$ we get

$$dz_1(\sigma) = z_1(\partial\sigma) = z(\partial\sigma) = dz(\sigma) = 0.$$

□

Definition 2.9 (Good cover). An open cover $(U_i)_{i \in I}$ of a topological space X is called *good* iff all finite intersections $U_{i_0} \cap \dots \cap U_{i_k}$ are either empty or contractible.

Remark 2.10. If X is a manifold it has a good cover given by geodesically convex balls. This also shows that every open cover can be refined by a good one.

Lemma 2.11. Let X be a topological space and $V_1, V_2 \subseteq X$ closed subsets covering X . For any open cover $\alpha = (U_i)_{i \in I}$ of X the corresponding nerves satisfy

$$X_\alpha = (V_1)_\alpha|_{V_1} \cup (V_2)_\alpha|_{V_2}.$$

Moreover we have

$$(V_1 \cap V_2)_\alpha|_{V_1 \cap V_2} \subseteq (V_1)_\alpha|_{V_1} \cap (V_2)_\alpha|_{V_2} \quad (2.3)$$

and if α is a good cover we have equality in (2.3).

Proof. Any q -simplex of X_α corresponds to a finite nonempty intersection $U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset$ and this yields a nonempty intersection $U_{i_0} \cap \dots \cap U_{i_q} \cap V_1 \neq \emptyset$ or $U_{i_0} \cap \dots \cap U_{i_q} \cap V_2 \neq \emptyset$ proving the first equality.

Every q -simplex of $(V_1 \cap V_2)_\alpha$ corresponds to a nonempty intersection $U_{i_0} \cap \dots \cap U_{i_q} \cap V_1 \cap V_2 \neq \emptyset$. Hence both $U_{i_0} \cap \dots \cap U_{i_q} \cap V_1$ and $U_{i_0} \cap \dots \cap U_{i_q} \cap V_2$ are nonempty proving the inclusion (2.3).

Let α be a good cover and assume there exists a q -simplex $\sigma = \{i_0, \dots, i_q\}$ in

$$((V_1)_\alpha \cap (V_2)_\alpha) \setminus (V_1 \cap V_2)_\alpha$$

and denote $U_\sigma = U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset$. Since α is good U_σ is contractible hence connected.

Since σ is not a simplex of $(V_1 \cap V_2)_\alpha$ we have $U_\sigma \cap V_1 \cap V_2 = \emptyset$. Because it is a simplex of both $(V_1)_\alpha$ and $(V_2)_\alpha$ we have $U_\sigma \cap V_j \neq \emptyset$ ($j = 1, 2$). Thus the $U_\sigma \cap V_j$ ($j = 1, 2$) constitute a decomposition of U_σ into closed, disjoint and nonempty subsets. This contradicts the connectedness of U_σ . □

Proof of Proposition 2.3. Let us prove the additivity of $f_*\mu_X$ assuming we have already proven the one of μ_X . The preimages $f^{-1}U_1$ and $f^{-1}U_2$ are still disjoint and open and the additivity of μ_X implies

$$f_*\mu_X(U_1 \dot{\cup} U_2) = \mu_X(f^{-1}U_1 \dot{\cup} f^{-1}U_2) = \mu_X(f^{-1}U_1) + \mu_X(f^{-1}U_2) = f_*\mu_X(U_1) + f_*\mu_X(U_2).$$

So it remains to prove the additivity of μ_X .

Consider the complements $V_i := X \setminus U_i$ ($i = 1, 2$). These are closed and cover X . The only inclusion which does not follow from monotonicity is

$$\mu(U_1 \dot{\cup} U_2) \subseteq \mu(U_1) + \mu(U_2).$$

Let $[z] \in \mu_X(U_1 \dot{\cup} U_2) \subseteq \check{H}^q X$, i.e. $[z]$ is a Čech cohomology class such that

$$[z]|_{V_1 \cap V_2} = 0 \in \check{H}^q(V_1 \cap V_2).$$

We have to show that $[z]$ can be written as a sum $[z] = [z_1] + [z_2]$ ($[z_i] \in \check{H}^q X$) such that $[z_i]|_{V_i} = 0$ ($i = 1, 2$).

By Lemma 2.6 we can assume that there exists an open cover δ of X such that $[z] \in \check{H}^q X = \varinjlim_{\alpha} H^q X_{\alpha}$ can be represented by a cohomology class $[z]_{\delta} \in H^q X_{\delta}$ and $[z]_{\delta}|_{(V_{\delta|V})} = 0 \in H^q V_{\delta|V}$. Using Remark 2.10 we can further refine δ and assume that it is a good cover. Now we get rid of the direct systems and their limits and we can restrict ourselves to the case of ordinary, simplicial cohomology groups.

Consider the subcomplexes $L_i := (V_i)_{\delta|V_i} \subseteq X_{\delta}$ ($i = 1, 2$). By Lemma 2.11 we have $X_{\delta} = L_1 \cup L_2$ and

$$(V_1 \cap V_2)_{\delta|V_1 \cap V_2} = L_1 \cap L_2. \quad (2.4)$$

With this notation (2.1) becomes $[z]_{\delta}|_{L_1 \cap L_2} = 0 \in H^q(L_1 \cap L_2)$. Using Lemma 2.8 there exist $[z_i] \in H^q X_{\delta}$ ($i = 1, 2$) with $[z] = [z_1] + [z_2]$ and $[z_i]|_{L_i} = 0 \in H^q L_i$ ($i = 1, 2$). The classes $[z_i] \in H^q X_{\delta}$ descend to the desired elements in $\check{H}^q X$. \square

Remark 2.12. (i) One reason why we always insist on using Čech instead of singular cohomology is that we do not know whether additivity in this generality holds for the latter.

(ii) We also do not know if X really needs to be a manifold. Otherwise we could not refine the open cover δ such that it becomes good. We would not have equality (but a proper inclusion) in equation (2.4) and we could not apply Lemma 2.8.

Proposition 2.13 (Multiplicativity). Let X be topological space and $(A, \smile) := (\check{H}^* X, \smile)$ its Čech cohomology algebra where \smile denotes the cup product. The standard $\mathcal{I}(A)$ -valued measure μ_X on X satisfies *multiplicativity*, i.e. for any two open subsets $U_1, U_2 \subseteq X$ we have

$$\mu_X(U_1) \smile \mu_X(U_2) \subseteq \mu_X(U_1 \cap U_2) \quad (2.5)$$

where the left hand side is meant as the product of ideals.

The analogous statement also holds for the pushforward measure $f_*\mu_X$ along any continuous map $f: X \rightarrow Y$.

Proof. Let us prove the multiplicativity of $f_*\mu_X$ assuming we have already proven the one of μ_X . The preimages $f^{-1}U_1$ and $f^{-1}U_2$ are still open and the multiplicativity of μ_X implies

$$\begin{aligned} f_*\mu_X(U_1) \smile f_*\mu_X(U_2) &= \mu_X(f^{-1}U_1) \smile \mu_X(f^{-1}U_2) \\ &\subseteq \mu_X(f^{-1}U_1 \cap f^{-1}U_2) = \mu_X(f^{-1}(U_1 \cap U_2)) = f_*\mu_X(U_1 \cap U_2). \end{aligned}$$

So it remains to prove the multiplicativity of μ_X .

Consider the complements $V_i := X \setminus U_i$ ($i = 1, 2$). The left hand side of (2.5) is additively generated by products of the form $[x_1] \smile [x_2]$ where $[x_i] \in \mu_X(U_i) \subseteq \check{H}^*X$, i.e. $[x_i]|_{V_i} = 0 \in \check{H}^{q_i}V_i$ ($i = 1, 2$). Using Lemma 2.6 there exists an open cover δ of X such that each $[x_i] \in \check{H}^{q_i}X = \varinjlim_{\alpha} H^{q_i}X_{\alpha}$ is represented by some cohomology class $[x_i]_{\delta} \in H^{q_i}X_{\delta}$ and for $i = 1, 2$ the restriction homomorphisms satisfy

$$\begin{aligned} H^{q_i}X_{\delta} &\rightarrow H^{q_i}(V_i)_{\delta|V_i} \\ [x_i]_{\delta} &\mapsto 0. \end{aligned}$$

Using Lemma 2.7 we can assume that the classes $[x_i]_{\delta}$ are represented by simplicial q_i -cocycles $(x_i)_{\delta} \in C^{q_i}X_{\delta}$ satisfying $(x_i)_{\delta}|_{(V_i)_{\delta|V_i}} = 0 \in C^{q_i}(V_i)_{\delta|V_i}$. Let $q := q_1 + q_2$.

Now in this simple case the cup product $[x_1] \smile [x_2] \in \check{H}^qX$ in Čech cohomology is represented by $[x_1]_{\delta} \smile [x_2]_{\delta} \in H^qX_{\delta}$ as well as the restriction $[x_1] \smile [x_2]|_{V_1 \cup V_2} \in \check{H}^q(V_1 \cup V_2)$ is represented by $[x_1]_{\delta} \smile [x_2]_{\delta}|_{(V_1 \cup V_2)_{\delta|V_1 \cup V_2}} \in H^q(V_1 \cup V_2)_{\delta}$ and we have

$$[x_1]_{\delta} \smile [x_2]_{\delta}|_{(V_1 \cup V_2)_{\delta|V_1 \cup V_2}} \quad (2.6)$$

$$= [(x_1)_{\delta} \smile (x_2)_{\delta}]|_{(V_1 \cup V_2)_{\delta|V_1 \cup V_2}} \quad (2.7)$$

$$= \left[(x_1)_{\delta} \smile (x_2)_{\delta} \right]_{(V_1 \cup V_2)_{\delta|V_1 \cup V_2}}. \quad (2.8)$$

From Lemma 2.11 we know that

$$(V_1 \cup V_2)_{\delta|V_1 \cup V_2} = (V_1)_{\delta|V_1} \cup (V_2)_{\delta|V_2}. \quad (2.9)$$

If the open cover δ is given by $(W_i)_{i \in I}$ then I is the vertex set of $(V_1 \cup V_2)_{\delta|V_1 \cup V_2}$. Fix a strict total ordering on this vertex set I . As usual one constructs the simplicial chain complex and dual to it the simplicial cochain complex. We need this in order to (simply) define the simplicial cup product. Let $\sigma := (v_0 < \dots < v_q)$ be an arbitrary q -simplex in $(V_1 \cup V_2)_{\delta|V_1 \cup V_2}$. Without loss of generality (cf. (2.9)) it is a simplex of $(V_1)_{\delta|V_1}$. The simplicial cup product is defined by

$$(x_1)_{\delta} \smile (x_2)_{\delta}(\sigma) = (x_1)_{\delta}(v_0 < \dots < v_{q_1}) \cdot (x_2)_{\delta}(v_{q_1+1} < \dots < v_{q_1+q_2})$$

and the first factor vanishes because of $(x_1)_{\delta}|_{(V_1)_{\delta|V_1}} = 0$. This proves that the expression (2.8) vanishes and hence $[x_1] \smile [x_2]|_{V_1 \cup V_2} = 0 \in \check{H}^q(V_1 \cup V_2)$. \square

Proposition 2.14 (Continuity). Let X be a compact topological space and $A := \check{H}^*X$ its Čech cohomology algebra. The standard ideal valued measure μ_X on X satisfies *continuity*, i.e. for any increasing nested sequence of open subsets $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots \subseteq X$ we have

$$\mu \left(\bigcup_{i=1}^{\infty} U_i \right) = \bigcup_{i=1}^{\infty} \mu(U_i). \quad (2.10)$$

The analogous statement also holds for the pushforward measure $f_*\mu_X$ along any continuous map $f: X \rightarrow Y$.

We will reduce this proposition to the so-called *continuity* of Čech cohomology. In order to state this property properly we need a little preparation.

Definition 2.15. A *compact pair* (X, A) is a pair of spaces such that X is compact and $A \subseteq X$ is closed. In particular A itself is compact. Let Z be a topological space. A sequence of pairs $(X_i, A_i) \subseteq (Z, Z)$ ($i \in \mathbb{N}$) together with inclusions $\iota_i^j: (X_i, A_i) \hookrightarrow (X_j, A_j)$ whenever $i < j$ is called a *nested sequence of pairs in Z* and we denote it by $((X_i, A_i)_{i \in \mathbb{N}}, \iota_i^j)$. For such a nested sequence its *intersection* is the topological pair $(X, A) \subseteq (Z, Z)$ defined by $X := \bigcap_i X_i$ and $A := \bigcap_i A_i$.

We will only need the following very weak version of continuity.

Theorem 2.16 (Continuity of Čech cohomology, [4, Theorem 2.6]). Let (X, A) be the intersection of a nested sequence of compact pairs. Let $\iota_i: (X, A) \hookrightarrow (X_i, A_i)$ denote the inclusion. Each $u \in \check{H}^q(X, A)$ is of the form $\iota_i^* u_i$ for some $i \in \mathbb{N}$ and some $u_i \in \check{H}^q(X_i, A_i)$.

Proof of Proposition 2.14. Let us prove the continuity of $f_*\mu_X$ assuming we have proven the one of μ_X . The subsets $(f^{-1}U_i)_{i \in \mathbb{N}}$ form an increasing nested sequence of open subsets of X and the continuity of μ_X implies

$$f_*\mu_X \left(\bigcup_{i=1}^{\infty} U_i \right) = \mu_X \left(f^{-1} \bigcup_{i=1}^{\infty} U_i \right) = \mu_X \left(\bigcup_{i=1}^{\infty} f^{-1}U_i \right) = \bigcup_{i=1}^{\infty} \mu_X(f^{-1}U_i) = \bigcup_{i=1}^{\infty} f_*\mu_X(U_i).$$

It remains to prove the continuity of μ_X .

Consider the complements $V_i := X \setminus U_i$ ($i \in \mathbb{N}$) and $V = \bigcap_i V_i = X \setminus \bigcup_i U_i$. The only inclusion of (2.10) not following from monotonicity is

$$\mu \left(\bigcup_{i=1}^{\infty} U_i \right) \subseteq \bigcup_{i=1}^{\infty} \mu(U_i),$$

i.e. given a cohomology class $[x] \in \check{H}^q X$ satisfying $[z]|V = 0 \in \check{H}^q V$ we have to show the existence of an index $i \in \mathbb{N}$ such that $[z]|V_i = 0$.

Consider the nested sequence of compact pairs given by $(X, V_i)_{i \in \mathbb{N}}$. The intersection of this nested sequence is precisely (X, V) . For every $i \in \mathbb{N}$ naturality of the long exact sequence yields the following commutative diagram.

$$\begin{array}{ccccc} \check{H}^q(X, V_i) & \longrightarrow & \check{H}^q X & \longrightarrow & \check{H}^q V_i \\ \downarrow & & \parallel & & \downarrow \\ \check{H}^q(X, V) & \longrightarrow & \check{H}^q X & \longrightarrow & \check{H}^q V \end{array}$$

In the diagram above every arrow is given by restriction. Because the class $[z] \in \check{H}^q X$ satisfies $[z]|V = 0 \in \check{H}^q V$ we can lift $[z]$ to a class $[\tilde{z}] \in \check{H}^q(X, V)$. By Theorem 2.16 there exists an index $i \in \mathbb{N}$ and a class $[u_i] \in \check{H}^q(X, V_i)$ such that $[u_i]|(X, V) = [\tilde{z}]$. We get $[u_i]|X = [z]$ and the top horizontal exact sequence yields $[z]|V_i = 0$. \square

Remark 2.17. (i) The continuity axiom fails if X is not compact. Let $X = B^2 \setminus 0$ and $V_i := \{x \in X \mid x_1 \leq \frac{1}{i}\}$. The intersection $\bigcap_i V_i = \{x \in X \mid x_1 \leq 0\}$ is contractible but the generator of $\check{H}^1 X$ survives when restricted to any V_i .
(ii) Continuity is the second reason why we prefer Čech over singular cohomology.

This motivates the following

Definition 2.18 (Ideal valued measures, [9, Section 4.1]). Let Y be a topological space, τ_Y the system of open subsets of Y , $A = \bigoplus_{n=0}^{\infty} A^n$ a graded commutative R -algebra and $\mathcal{I}(A)$ the set of graded ideals $I \subseteq A$. An $\mathcal{I}(A)$ -valued measure μ on Y is a map

$$\mu: \tau_Y \rightarrow \mathcal{I}(A)$$

assigning a graded ideal $\mu(U) \subseteq A$ to any open $U \subseteq Y$ such that the following properties hold:

- (i) Normalisation: $\mu(\emptyset) = 0$.
- (ii) Monotonicity: For $U_1 \subseteq U_2$ we have $\mu(U_1) \subseteq \mu(U_2)$.
- (iii) Continuity: For any increasing nested sequence $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ we have

$$\mu\left(\bigcup_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} \mu(U_i).$$

- (iv) Additivity: For two disjoint open subsets we have

$$\mu(U_1 \dot{\cup} U_2) = \mu(U_1) + \mu(U_2).$$

- (v) Multiplicativity: We have

$$\mu(U_1) \cdot \mu(U_2) \subseteq \mu(U_1 \cap U_2)$$

for any open $U_1, U_2 \subseteq Y$.

- (vi) Fullness: We have $\mu(Y) = A$.

The main instances of the definition above are given in the following

Corollary 2.19. Let X be a compact manifold and $A := \check{H}^* X$ its Čech cohomology algebra. The standard $\mathcal{I}(A)$ -valued measure as well as the pushforward measure $f_* \mu_X$ along any continuous map $f: X \rightarrow Y$ are ideal valued measures in the sense of the preceding definition.

Proof. This is an immediate consequence of Definition 2.1, Propositions 2.3, 2.13 and 2.14. \square

Remark 2.20. (i) Given an ideal valued measure μ on Y it will turn out to be useful to define the *vanishing ideal*

$$\mathbf{0}_\mu(A) := \mu(X \setminus A)$$

for any closed $A \subseteq Y$. If the measure is clear from the context, mostly the standard measure or its pushforward along a continuous map, we will only write $\mathbf{0}(A)$.

- (ii) In [9] there is one more axiom. By definition an ideal valued measure satisfies the *intersection property* iff for any two open $U_1, U_2 \subseteq Y$ covering Y we have $\mu(U_1 \cap U_2) = \mu(U_1) \cap \mu(U_2)$. One can show that the standard measure and every pushforward of it satisfy this intersection property but we will not need this for our applications.

Let X and Y be topological spaces and R a coefficient ring such that the rank of a homomorphism between R -modules makes sense, e.g. \mathbb{Z} , \mathbb{Z}_2 or \mathbb{Q} .

Definition 2.21 (Waist functionals). Recall from Definition 1.3 that for every continuous map $f: X \rightarrow Y$ the *total* or *degree k cohomological width* of f is given by

$$\begin{aligned} \text{width}_*(f) &:= \max_{y \in Y} \text{rk} [\check{H}^* X \rightarrow \check{H}^* f^{-1}y] \quad \text{and} \\ \text{width}_k(f) &:= \max_{y \in Y} \text{rk} [\check{H}^k X \rightarrow \check{H}^k f^{-1}y]. \end{aligned}$$

They give rise to the *waist functionals* width_* and width_k both of which are (not necessarily in any sense continuous) maps $C(X, Y) \rightarrow \mathbb{N}_0$ where $C(X, Y)$ is the space of all continuous maps $f: X \rightarrow Y$.

In this paper we will give lower bounds of $\text{width}_k(X/Y)$ for various fixed manifolds X and Y . These proofs of these at first only work for generic maps $f: X \rightarrow Y$, e.g. we will find a lower bound of $\text{width}_k(f)$ for all smooth f which intersect some smooth triangulation of Y transversally. In this section we will show that the same lower bound will also hold for all continuous f . In other words it is sufficient to prove waist inequalities just for (in some sense) generic maps. This is motivated by a sentence in [9, p. 417] about a quantity which “may only *increase* under uniform limits of maps”. The aim of this section is to render this precise.

Proposition 2.22 (Upper semi-continuity of waists). Let X and Y be compact and Y metrisable. If the Čech cohomology algebra $\check{H}^* X$ is finite dimensional the waist functionals width_* and $\text{width}_k: C(X, Y) \rightarrow \mathbb{N}_0$ are upper semi-continuous with respect to the compact-open topology.

Proof. We will just show the upper semi-continuity of width_* . The corresponding statement for width_k can be proven analogously. Endow Y with an arbitrary metric d . The compact-open topology is identical with the metric topology induced from the uniform norm. As $C(X, Y)$ is a metric space semi-continuity is equivalent to sequential semi-continuity. So given a sequence of functions $f_n: X \rightarrow Y$ uniformly converging to f we need to show that $\text{width}_*(f_n) \geq \alpha$ for every n implies

$$\text{width}_*(f) \geq \alpha.$$

Hence for every n there exists a point $y_n \in Y$ such that

$$\text{rk} [\check{H}^* X \rightarrow \check{H}^* f_n^{-1}y_n] = \text{rk} \left[\check{H}^* X \Big/ (f_n)_* \mu_X(Y \setminus y_n) \right] \geq \alpha$$

where $(f_n)_* \mu_X$ is the pushforward of the standard ideal valued measure on X and we used Corollary 2.2.

Since Y is sequentially compact we can pass to a subsequence and assume that the y_n converge to some point $y \in Y$ and that the convergences $f_n \rightarrow f$ and $y_n \rightarrow y$ are controlled

by

$$\begin{aligned} d(y_n, y) &< \frac{1}{8n^2} \\ d(f_n, f) &< \frac{1}{8n^2}. \end{aligned}$$

We claim the following equality of subsets of X .

$$\bigcup_{n>0} \left\{ x \in X \mid d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} = \{f(-) \neq y\} \quad (2.11)$$

Let us first discuss the inclusion “ \subseteq ”: For every $x \in X$ with $d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n}$ for some n the reverse triangle inequality implies

$$\begin{aligned} d(f(x), y) &\geq d(f_n(x), y_n) - d(f_n(x), f(x)) - d(y_n, y) \\ &> \frac{1}{4n^2} + \frac{1}{n} - \frac{1}{8n^2} - \frac{1}{8n^2} = \frac{1}{n} > 0 \end{aligned}$$

Similarly the inclusion “ \supseteq ” can be shown as follows: If $x \in X$ satisfies $d(f_n(x), y_n) \leq \frac{1}{4n^2} + \frac{1}{n}$ for every n we can conclude

$$\begin{aligned} d(f(x), y) &\leq d(f(x), f_n(x)) + d(f_n(x), y_n) + d(y_n, y) \\ &< \frac{1}{8n^2} + \frac{1}{4n^2} + \frac{1}{n} + \frac{1}{8n^2} \rightarrow 0 \end{aligned}$$

and hence $f(x) = y$. This proves (2.11).

Moreover we claim that the sets on the left hand side of (2.11) are nested, i.e. we have

$$\left\{ x \in X \mid d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} \subseteq \left\{ x \in X \mid d(f_{n+1}(x), y_{n+1}) > \frac{1}{4(n+1)^2} + \frac{1}{n+1} \right\}. \quad (2.12)$$

If $x \in X$ is an element of the left hand side we have

$$\begin{aligned} d(f_{n+1}(x), y_{n+1}) &> d(f_n(x), y_n) - d(f_n(x), f_{n+1}(x)) - d(y_n, y_{n+1}) \\ &> \frac{1}{4n^2} + \frac{1}{n} - \frac{1}{4n^2} - \frac{1}{4n^2} = \frac{1}{n} - \frac{1}{4n^2} > \frac{1}{4(n+1)^2} + \frac{1}{n+1}. \end{aligned}$$

proving (2.12).

The continuity axiom (which holds by Proposition 2.14 since X is compact) implies

$$\bigcup_{n>0} f_*\mu_X \left\{ x \in X \mid d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} = f_*\mu_X \{x \in X \mid f(x) \neq y\}.$$

The left hand side is an increasing sequence of ideals in \check{H}^*X and since the latter is finite dimensional there exists an $n > 0$ such that

$$f_*\mu_X \left\{ x \in X \mid d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} = f_*\mu_X \{x \in X \mid f(x) \neq y\}.$$

Monotonicity yields

$$\{x \in X \mid f_n(x) \neq y_n\} \supseteq \{x \in X \mid f(x) \neq y\}$$

proving

$$\mathrm{rk} \left[\check{H}^* X / f_* \mu_X(Y \setminus y) \right] \geq \mathrm{rk} \left[\check{H}^* X / (f_n)_* \mu_X(Y \setminus y_n) \right] \geq \alpha. \quad \square$$

- Remark 2.23.** (i) The waist functionals fail to be lower semi-continuous. Consider the embedding $g: S^2 \hookrightarrow D^3$ and the sequence $f_n: S^2 \hookrightarrow D^3$ shrinking g to a point, e.g. $f_n(x) = g(x)/n$. This sequence uniformly converges to the constant map f with value $0 \in D^3$ but $\mathrm{width}_2(f_n) = 0$ whereas $\mathrm{width}_2(f) = 1$.
- (ii) One question which immediately arises about the definition of cohomological width of a map $f: X \rightarrow Y$ is why we defined it as

$$\mathrm{width}_k(f) = \max_{y \in Y} \mathrm{rk} [H^k X \rightarrow H^k f^{-1}y]$$

where we could have equally been interested in

$$w_k(f) := \max_{y \in Y} \mathrm{rk} H^k f^{-1}y.$$

However this functional $w_k: C(X, Y) \rightarrow \mathbb{N}_0$ fails to be upper semi-continuous. Consider the composition

$$g: S^2 \hookrightarrow D^3 \rightarrow [-1, 1]$$

where the first map is the standard embedding and the second map is the restriction of a linear projection, e.g. onto the x -axis. Again we consider the family $f_n(x) := g(x)/n$ which converges uniformly to f , the constant map with value $0 \in [0, 1]$. All fibers of f_n are points or circles so we have $w_1(f_n) = 1$ but the only nonempty fiber of f is S^2 hence $w_1(f) = 0$.

Nevertheless we clearly have $w_k(f) \geq \mathrm{width}_k(f)$ so any lower bound for $\mathrm{width}_k(f)$ is also one for $w_k(f)$.

- (iii) The proof of Proposition 2.22 still works if one weakens the assumption that $\check{H}^* X$ is finite dimensional to $\check{H}^* X$ being finitely generated as an algebra since all finitely generated graded commutative algebras over Noetherian base rings are Noetherian.

3. FILLING ARGUMENT

Prior to prove cohomological waist inequalities we need some preliminaries. We will deal with various kinds of manifolds such as smooth manifolds, topological manifolds and manifolds with corners (cf. [12]). If any specifier is missing by a manifold we mean a smooth manifold. An example of a smooth manifold with corners up to codimension k is the standard k -simplex Δ^k . Another source of examples will given in the following proposition.

Definition 3.1 (Smooth, embedded simplices). Let N^q be a manifold. A *smooth, embedded k -simplex σ in N* is a smooth map $\sigma: \Delta^k \rightarrow N$ such that there exists an open neighbourhood $\Delta^k \subset U \subset \mathbb{R}^k$ and a smooth extension $\tilde{\sigma}: U \rightarrow N$ which is an embedding.

Definition 3.2 (Stratum transversality). Let M^n and N^q be manifolds without boundary, $f: M \rightarrow N$ smooth and $\sigma: \Delta^k \rightarrow N$ a smooth, embedded simplex. We say that f intersects σ *stratum transversally* if f intersects the interior of σ transversally and all of its faces stratum transversally. The map f intersects a 0-simplex stratum transversally iff its image point is a regular value of f .

Proposition 3.3 (Generic preimages of simplices). Let M^n and N^q be closed, oriented manifolds, $\sigma: \Delta^k \rightarrow N$ a smooth, embedded simplex and $f: M \rightarrow N$ a smooth map intersecting σ stratum transversally.

The preimage $f^{-1}\sigma(\Delta^k)$ is an oriented topological $(n - q + k)$ -manifold with boundary

$$\partial f^{-1}\sigma(\Delta^k) = f^{-1}\sigma(\partial\Delta^k).$$

Proof. Theorem 3 in [14] shows that $f^{-1}\sigma(\Delta^k)$ is a smooth manifold with corners up to codimension k hence it is a topological manifold with boundary. Note that most of the technical assumptions are met since M and N do not have boundary. Moreover the theorem states that the codimension l corner points of $f^{-1}\sigma(\Delta^k)$ are precisely the preimages of codimension l corner points of Δ^k , in particular $\partial f^{-1}\sigma(\Delta^k) = f^{-1}\sigma(\partial\Delta^k)$. \square

Mind the following notational convention.

Notation 3.4. In the situation of Proposition 3.3 we frequently denote the preimage of a simplex $\sigma: \Delta^k \rightarrow N$ by

$$F_\sigma := f^{-1}\sigma(\Delta^k)$$

and similarly

$$F_{\partial\sigma} := f^{-1}\sigma(\partial\Delta^k) = \partial F_\sigma.$$

We will often use this notation without explicitly mentioning it.

Definition 3.5 (Smooth triangulations). Let N^q be a smooth manifold. A *smooth triangulation* $\mathcal{T} = (K, \varphi)$ of N consists of a finite simplicial complex K together with a homeomorphism $\varphi: |K| \rightarrow N$ such that the restriction of φ to any simplex yields a smooth, embedded simplex in N . The set of all of these smooth k -simplices of \mathcal{T} shall be denoted by \mathcal{T}_k . We will often omit the specification *smooth* and simply talk about *a triangulation* and its *simplices*.

If N^q is R -oriented a triangulation \mathcal{T} is called *R -oriented* iff the sum of the elements in \mathcal{T}_q , i.e. the top-dimensional simplices, represents the R -oriented fundamental class of N^q .

Any smooth manifold N admits a smooth triangulation [13, Theorem 10.6].

Proposition 3.6. Let $f: M^n \rightarrow N^q$ be a smooth map between closed R -oriented manifolds, \mathcal{T} an R -oriented triangulation of N such that f intersects all the simplices $\sigma \in \mathcal{T}_q$ stratum transversally. For $k = 0, \dots, q$ we can inductively assign singular chains $c_\sigma \in C_{n-q+k}(F_\sigma; R)$ to every $\sigma \in \mathcal{T}_k$ such that the following properties hold.

- (i) For $\sigma \in \mathcal{T}_0$ the chain $c_\sigma \in C_{n-q}(F_\sigma; R)$ represents the (correctly oriented) fundamental class of F_σ .
- (ii) For $1 \leq k \leq q$ and $\sigma \in \mathcal{T}_k$ we can view the sum

$$\sum_{i=0}^k (-1)^i c_{\partial_i \sigma} \tag{3.1}$$

as an element of $C_{n-q+k-1}(\partial F_\sigma; R)$ and this represents the (correctly oriented) fundamental class of ∂F_σ with the boundary orientation. The element $c_\sigma \in C_{n-q+k}(F_\sigma; R)$

satisfies

$$\partial c_\sigma = \sum_{i=0}^k (-1)^i c_{\partial_i \sigma} \quad (3.2)$$

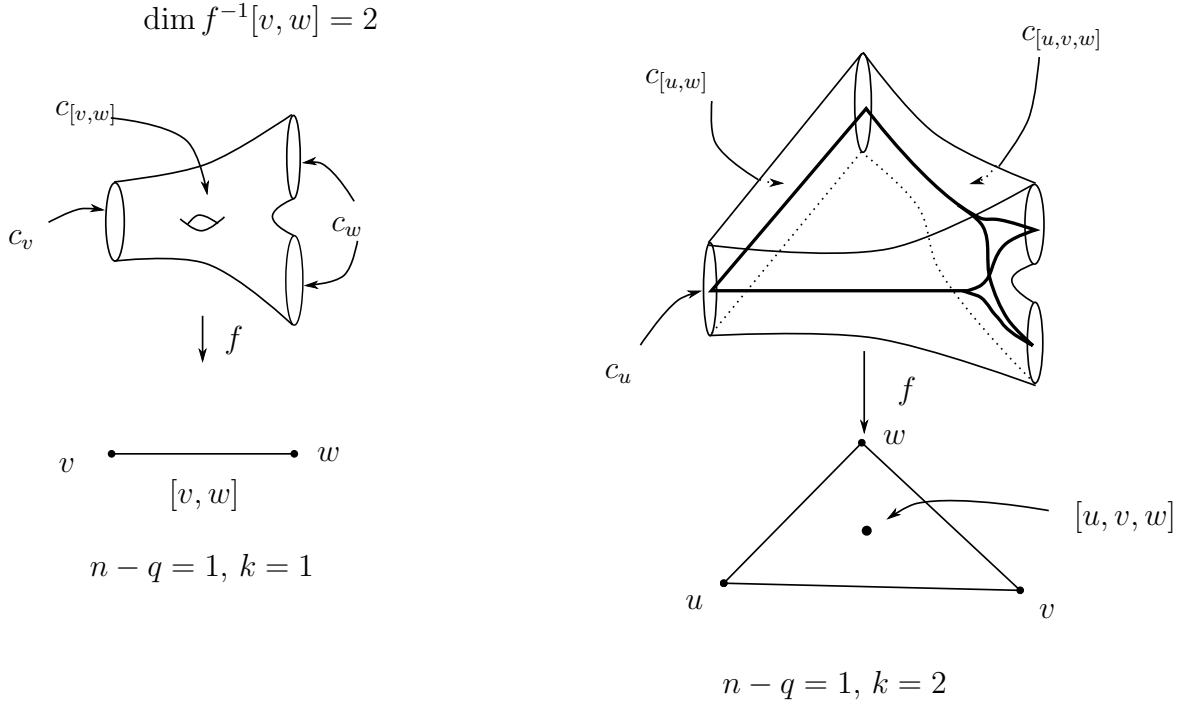
as an equation in $C_{n-q+k-1}(F_\sigma; R)$ and c_σ represents the (correctly oriented) relative fundamental class in $H_{n-q+k}(F_\sigma, \partial F_\sigma; R)$.

(iii) The sum

$$\sum_{\sigma \in \mathcal{T}_q} c_\sigma \in C_n(M; R) \quad (3.3)$$

represents the (correctly oriented) fundamental class of M .

$$\dim f^{-1}[u, v, w] = 3$$



In the exemplary picture on the right hand side $c_{[u, w]}$ is a cylinder and both $c_{[u, v]}$ and $c_{[v, w]}$ are pairs of pants. The chain $c_{[u, v, w]}$ is a solid torus. The bold line is mapped to the barycentre of $[u, v, w]$ and the farther a point in $c_{[u, v, w]}$ is from this core line the closer it is mapped to $\partial[u, v, w]$.

Remark 3.7. Technically the summands appearing in the expressions (3.1), (3.2) and (3.3) are elements of different chain groups $C_{n-q+k-1}F_{\partial_i \sigma}$ (for varying i) or $C_n F_\sigma$ (for varying σ). In order to make sense of the sums and equations we view these summands as chains in the chain group of the larger space $F_{\partial \sigma}$ or M . For the sake of legibility we omit to denote all the inclusions and their induced maps on chain groups and ask the reader to interpret such equations of cycles in a sensible way. This convention holds for the rest of this paper.

Proof of Proposition 3.6. Proposition 3.3 shows that for all $\sigma \in \mathcal{T}_k$ the preimage F_σ is an oriented topological $(n-q+k)$ -manifold with boundary $F_{\partial\sigma}$. Hence the notion of fundamental classes makes sense. Bear in mind that both F_σ and ∂F_σ may be empty or have several components.

- (i) For every $\sigma \in \mathcal{T}_0$ the preimage F_σ is a closed oriented $(n-q)$ -dimensional submanifold of M and it is easy to arrange (i). We proceed by induction over k and assume that we have constructed chains c_τ for all simplices $\tau \in \mathcal{T}_l$ of dimension $l < k$.
- (ii) A standard calculation shows

$$\partial \sum_{i=0}^k (-1)^i c_{\partial_i \sigma} = \sum_{i=0}^k (-1)^i \partial c_{\partial_i \sigma} = \sum_{i=0}^k (-1)^i \sum_{j=0}^{k-1} (-1)^j c_{\partial_j \partial_i \sigma} = 0.$$

Hence $\sum_{i=0}^k (-1)^i c_{\partial_i \sigma}$ defines a homology class in $H_{n-q+k-1}(\partial F_\sigma)$. For every $0 \leq j \leq k$ the induced maps of the inclusions satisfy

$$\begin{aligned} H_{n-q+k-1}(\partial F_\sigma) &\rightarrow H_{n-q+k-1} \left(\partial F_\sigma, \bigcup_{i \neq j} F_{\partial_i \sigma} \right) \\ \left[\sum_{i=0}^k (-1)^i c_{\partial_i \sigma} \right] &\mapsto [(-1)^j c_{\partial_j \sigma}]. \end{aligned}$$

For every $p \in F_{\partial_j \sigma}$ the image of these classes in $H_{n-q+k-1}(F_\sigma, F_\sigma \setminus p)$ is the correct local orientation of $F_{\partial_j \sigma}$ in the point p where $F_{\partial_j \sigma} \subseteq \partial F_\sigma$ is oriented as the boundary of F_σ . This proves that $\sum_{i=0}^k (-1)^i c_{\partial_i \sigma}$ represents the (correctly oriented) fundamental class of $F_{\partial\sigma}$.

The fundamental class $[c_\sigma] \in H_{n-q+k}(F_\sigma, \partial F_\sigma)$ satisfies

$$\partial: H_{n-q+k}(F_\sigma, \partial F_\sigma) \rightarrow H_{n-q+k-1}(\partial F_\sigma) \quad (3.4)$$

$$[c_\sigma] \mapsto \left[\sum_{i=0}^k (-1)^i c_{\partial_i \sigma} \right] \quad (3.5)$$

and the relative cycle c_σ can be modified so as to achieve equation (3.2) on chain level.

- (iii) We have

$$\partial \sum_{\sigma \in \mathcal{T}_q} c_\sigma = \sum_{\sigma \in \mathcal{T}_q} \sum_{i=0}^q (-1)^i c_{\partial_i \sigma} = 0$$

since every $(q-1)$ -simplex is the face of exactly two q -simplices and inherits different orientations from them. Hence $\sum_{\sigma \in \mathcal{T}_q} c_\sigma$ defines a homology class in $H_n M$. Again for

every $\tau \in \mathcal{T}_q$ the inclusion $(M, \emptyset) \rightarrow (M, \bigcup_{\sigma \in \mathcal{T} \setminus \tau} F_\sigma)$ satisfies

$$H_n(M) \rightarrow H_n \left(M, \bigcup_{\sigma \in \mathcal{T} \setminus \tau} F_\sigma \right)$$

$$\left[\sum_{\sigma \in \mathcal{T}_q} c_\sigma \right] \mapsto c_\tau$$

and for every $p \in F_\tau$ arbitrary the image of these classes in $H_n(M, M \setminus p)$ yields the correct local orientation of M in p . \square

The rest of this section is devoted to the formulation and proof of Proposition 3.9, a genericity result which for any map $f: M \rightarrow N$ guarantees the existence of a triangulation of the target manifold N which is (in a precise sense) generic and fine.

Repetition 3.8. Let M and N be manifolds without boundary. We will denote the space of all continuous maps $f: M \rightarrow N$ by $C^0(M, N)$ and it shall be equipped with the compact-open topology. If M is compact the subspace topology on $C^\infty(M, N) \subset C^0(M, N)$ is coarser than the weak C^∞ -topology.

Proposition 3.9. Let M and N be two closed manifolds. For every smooth map $f: M \rightarrow N$ and every open cover $\mathcal{U} = (U_i)_{i \in I}$ of N there exists a smooth triangulation \mathcal{T} of N and a sequence of smooth maps $f_n: M \rightarrow N$ uniformly converging to f such that the following properties hold:

- (i) Every map f_n intersects every simplex $\sigma \in \mathcal{T}_k$ stratum transversally.
- (ii) For every $\sigma \in \mathcal{T}_k$ there exists an index $i \in I$ such that $\sigma(\Delta^k) \subseteq U_i$.

Proof. Choose a smooth triangulation $\mathcal{T} = (K, \varphi)$ of N and consider the preimage $\varphi^{-1}\mathcal{U} := (\varphi^{-1}U_i)_{i \in I}$ which is an open cover of $|K|$. Since N is compact this open cover has a Lebesgue number with respect to some standard metric on $|K|$. After barycentric subdivision we can assume that every simplex of $|K|$ is contained in some $f^{-1}U_i$, i.e. its image is contained in U_i .

For every smooth, embedded simplex $\sigma: \Delta^k \rightarrow N$ the subset

$$\{f \in C^\infty(M, N) | f \pitchfork \text{im } \sigma\} \subseteq C^\infty(M, N)$$

is a residual in the weak C^∞ -topology, i.e. it is the countable intersection of open and dense subsets [11, Transversality Theorem 2.1]. Moreover the Baire category theorem applies to the weak C^∞ -topology, i.e. every residual set is dense. The set

$$\{g \in C^\infty(M, N) | \text{every simplex intersects } g \text{ stratum transversally}\}$$

$$= \bigcap_{\sigma \text{ simplex of } \mathcal{T}} \{g \in C^\infty(M, N) | \sigma \text{ intersects } g \text{ stratum transversally}\}$$

is the countable intersection of residual sets, hence itself residual and therefore dense. Since the compact-open topology is coarser than the weak C^∞ -topology the claim follows. \square

For the rest of this paper N^q always denotes a smooth q -manifold. At the beginning we allow N to be disconnected, to have non-empty boundary or to be non-compact. Let us recall Theorem 1.4 which holds for this general class of target manifolds. We will quickly see that we can restrict ourselves to the case where N is closed and connected.

Theorem 1.4. Every continuous map $f: T^n \rightarrow N^q$ admits a point $y \in N^q$ such that the rank of the restriction homomorphism satisfies

$$\mathrm{rk} [H^1(T^n; \mathbb{Z}) \rightarrow H^1(f^{-1}(y); \mathbb{Z})] \geq n - q.$$

Remark 3.10. (i) This inequality is non-vacuous only if $n > q$ which we will tacitly assume. Furthermore it shows $\mathrm{width}_1(T^n/N) \geq n - q$.

(ii) Let us assume for the moment that we have proven the theorem for closed connected N . We will explain how the theorem extends to manifolds which are possibly disconnected, non-compact or have non-empty boundary. Since T^n is connected we can restrict the target of f to the component which is hit. If N had boundary consider the inclusion $N \hookrightarrow D$ into the double D of N . Since D has no boundary we can apply the theorem to the composition

$$T^n \xrightarrow{f} N \hookrightarrow D$$

yielding the theorem for N .

If N is non-compact we choose a sequence $N_1 \subset N_2 \subset \dots \subset N$ such that each N_i is a smooth compact codimension 0 submanifold with boundary and $\bigcup_{i=1}^{\infty} \mathrm{int} N_i = N$ (such an exhaustion exists by a strong form of the Whitney embedding theorem where every (even non-compact) manifold can be embedded into some \mathbb{R}^N with closed image). Since $f(T^n)$ is compact it is contained in N_i for some $i \gg 0$, i.e. we can view f as a map $T^n \rightarrow N_i$ and we already deduced the theorem for compact manifolds with boundary. For the rest of this paper we will assume the target manifold N to be closed and connected.

The theorem will essentially follow from the following

Proposition 3.11. If N^q is closed there is no smooth map $f: T^n \rightarrow N$ together with a smooth triangulation \mathcal{T} of N such that the following two properties hold:

- (i) The smooth simplices of \mathcal{T} intersect f stratum transversally.
- (ii) For every $\sigma \in \mathcal{T}_k$ the preimage $F_\sigma := f^{-1}\sigma(\Delta^k)$ satisfies

$$\mathrm{rk} [H^1(T^n; \mathbb{Z}) \rightarrow H^1(F_\sigma; \mathbb{Z})] < n - q.$$

Proof of Theorem 1.4 assuming Proposition 3.11. Assume there is a continuous map $f: T^n \rightarrow N^q$ such that $\mathrm{width}_1(f) < n - q$. Since T^n is compact the standard ideal valued measure μ_{T^n} on T^n satisfies the continuity axiom. The same holds for the pushforward measure $f_*\mu_{T^n}$ which is a measure on N . Recall from Remark 1.2.20 (i) that the vanishing ideal associated to $f_*\mu_{T^n}$ is defined by

$$\mathbf{0}(A) := f_*\mu_{T^n}(N \setminus A) = \ker [H^*T^n \rightarrow H^*f^{-1}A]$$

for every closed subset $A \subseteq N$. Corollary 1.2.2 implies

$$\mathrm{rk} [H^1T^n \rightarrow H^1f^{-1}y] = \mathrm{rk} \left[H^1T^n / \mathbf{0}(y) \cap H^1T^n \right]$$

for every $y \in N$ and therefore the condition $\text{width}_1(f) < n - q$ translates into

$$\text{rk} \left[H^1 T^n / \mathbf{o}(y) \cap H^1 T^n \right] < n - q.$$

Choose an arbitrary metric on N . With respect to this metric we have

$$\bigcap_{m=1}^{\infty} \overline{B \left(y, \frac{1}{m} \right)} = \{y\}.$$

The continuity property of $f_* \mu_{T^n}$ (translated into the language of vanishing ideals) yields

$$\bigcup_{m=1}^{\infty} \mathbf{o} \left(\overline{B \left(y, \frac{1}{m} \right)} \right) = \mathbf{o}(\{y\}).$$

Since $H^* T^n$ is finite dimensional there exists an $m(y) \gg 0$ depending on y such that

$$\mathbf{o} \left(\overline{B \left(y, \frac{1}{m(y)} \right)} \right) = \mathbf{o}(\{y\}).$$

For every closed subset $A \subset B \left(y, \frac{1}{m(y)} \right)$ we have $\mathbf{o}(A) \supseteq \mathbf{o} \left(\overline{B \left(y, \frac{1}{m(y)} \right)} \right)$ and hence

$$\text{rk}[H^1 T^n \rightarrow H^1 A] = \text{rk} \left[H^1 T^n / \mathbf{o}(A) \cap H^1 T^n \right] \quad (3.6)$$

$$\leq \text{rk} \left[H^1 T^n / \mathbf{o} \left(\overline{B \left(y, \frac{1}{m(y)} \right)} \right) \cap H^1 T^n \right] = \text{rk} \left[H^1 T^n / \mathbf{o}(y) \cap H^1 T^n \right] < n - q. \quad (3.7)$$

Every continuous map f can be uniformly approximated by smooth maps g_m . Since M and N are compact and metrisable the upper semi-continuity of width_1 (cf. Proposition 2.2.22) implies $\text{width}(g_m) \leq \text{width}(f) < n - q$ for $m \gg 0$. So without loss of generality we can assume that f itself is smooth. Applying Proposition 3.9 to this smooth map $f: T^n \rightarrow N^q$ and the open cover $\left(B \left(y, \frac{1}{m(y)} \right) \right)_{y \in N}$ yields a smooth triangulation \mathcal{T} of N and a sequence of smooth maps $f_m: M \rightarrow N$ uniformly converging to f such that the following two properties hold:

- (i) Every map f_m intersects every simplex $\sigma \in \mathcal{T}_k$ stratum transversally.
- (ii) For every simplex $\sigma \in \mathcal{T}_k$ there exists a $y(\sigma) \in N$ such that

$$\sigma(\Delta^k) \subseteq B \left(y(\sigma), \frac{1}{m(y(\sigma))} \right).$$

Using estimate (3.7) we conclude that $F_\sigma := f^{-1} \sigma(\Delta^k)$ satisfies $\text{rk} [H^1 T^n \rightarrow H^1 F_\sigma] < n - q$. Similarly as before we can use the upper semi-continuity of width_1 (Proposition 2.2.22) to get $\text{width}_1(f_m) < n - q$ for $m \geq M$. The map f_M contradicts Proposition 3.11. \square

Remark 3.12. In the future whenever we want to prove a lower bound for cohomological waist we will reduce it to the proof of a statement similar to Proposition 3.11. We will not carry out this reduction in detail anymore.

Recall that Theorem 1.4 we are trying to prove is about a map $f: T^n \rightarrow N^q$ and its fibers $F_y := f^{-1}y$. Amongst others we want to apply the following statements to the inclusions of the fibers $f_y: F_y \hookrightarrow T^n$. The reader shall bear this example in mind.

Lemma 3.13 (Filling Lemma). Let $k: K \rightarrow T^n$ be continuous and $\text{rk } H^1(k; \mathbb{Z}) < n - q$. There exists a relative CW complex $(\text{Fill}(k), K)$ and an extension $\text{fill}(k): \text{Fill}(k) \rightarrow T^n$ such that the diagram

$$\begin{array}{ccc} & \text{Fill}(k) & \\ \iota \uparrow & \searrow \text{fill}(k) & \\ K & \xrightarrow{k} & T^n \end{array}$$

commutes and the following properties hold.

- (i) Up to homotopy $\text{Fill}(k)$ is the disjoint sum of a number of tori, one copy for each component of K , i.e.

$$\text{Fill}(k) \simeq T^{r_1} \amalg T^{r_2} \amalg \dots$$

and the dimensions satisfy $r_i < n - q$. In particular we have $H_{\geq n-q}(\text{Fill}(k); G) = 0$ and $H_{\geq n-q}(\iota; G) = 0$ for any abelian coefficient group G .

- (ii) $(\text{Fill}(k), K)$ is homologically 1-connected
- (iii) $\text{rk } H^1(\text{fill}(k); \mathbb{Z}) = \text{rk } H^1(k; \mathbb{Z})$

Before we prove the lemma we need an analysis of the discrepancy between cohomology and homology.

Remark 3.14. (i) For every continuous map $f: X \rightarrow Y$ we have $\text{rk } H_1(f; \mathbb{Z}) = \text{rk } H^1(f; \mathbb{Z})$.
(ii) If $H_1(f; \mathbb{Z})$ is an isomorphism so is $H^1(f; \mathbb{Z})$.

Proof. This follows from the universal coefficient theorem. □

We will frequently change our point of view between cohomology and homology and we will do so without further reference to the remark above.

Notation 3.15. From now on we will have to introduce a lot of spaces all of which come with reference maps to T^n . As with $f_y: F_y \rightarrow T^n$ these reference maps are denoted by the lower case letters corresponding to the upper case letters representing the spaces.

Proof of Filling Lemma 3.13. Let us first discuss the case where K is connected and let $r := \text{rk } H^1(k; \mathbb{Z})$. By the naturality of the Hurewicz homomorphism the following diagram commutes.

$$\begin{array}{ccc} \pi_1 K & \longrightarrow & \pi_1 T^n = \mathbb{Z}^n \\ \downarrow & & \downarrow \cong \\ H_1(K; \mathbb{Z}) & \longrightarrow & H_1(T^n; \mathbb{Z}) = \mathbb{Z}^n \end{array} \tag{3.8}$$

This proves that $\text{im } \pi_1 k \subseteq \mathbb{Z}^n$ is also a rank r subgroup. Consider a covering $T^r \times \mathbb{R}^{n-r} \rightarrow T^n$ corresponding to this subgroup. Hence there exists a lift $\tilde{k}: K \rightarrow T^r \times \mathbb{R}^{n-r}$ such that

$$\begin{array}{ccc} & T^r \times \mathbb{R}^{n-r} & \\ \tilde{k} \nearrow & \downarrow & \\ K & \xrightarrow{k} & T^n \end{array}$$

commutes. On the level of fundamental groups this turns into the following diagram.

$$\begin{array}{ccc} & \pi_1(T^r \times \mathbb{R}^{n-r}) & \\ \pi_1 \tilde{k} \nearrow & \downarrow & \\ \pi_1 K & \xrightarrow{\pi_1 k} & \pi_1 T^n \end{array}$$

where (by construction of the covering) the vertical arrow is the inclusion $\text{im } \pi_1 k \subset \mathbb{Z}^n$. Thus $\pi_1 \tilde{k}$ is obtained from $\pi_1 k$ by restricting the target to $\text{im } \pi_1 k$, in particular $\pi_1 \tilde{k}$ is surjective. Using the naturality of the Hurewicz homomorphism similar to (3.8) we conclude that $H_1(\tilde{k}; \mathbb{Z})$ is surjective.

We want to turn \tilde{k} into the inclusion of relative CW complex. Substitute $T^r \times \mathbb{R}^{n-r}$ by the mapping cylinder $M_{\tilde{k}}$ and choose a relative CW approximation $(\text{Fill}(k), K)$ of $(M_{\tilde{k}}, K)$, i.e. there is a weak homotopy equivalence $\text{Fill}(k) \xrightarrow{\simeq} M_{\tilde{k}}$ restricting to the identity on K . Define ι and $\text{fill}(k)$ as in the following diagram.

$$\begin{array}{ccccc} & & \text{fill}(k) & & \\ & \nearrow & \text{---} & \searrow & \\ \text{Fill}(k) & \xrightarrow{\simeq} & M_{\tilde{k}} & \longrightarrow & T^n \\ & \nwarrow \iota & \uparrow & \nearrow k & \\ & & K & & \end{array}$$

The induced map $H_*(\iota; \mathbb{Z})$ is an isomorphism for $* = 0$ and surjective for $* = 1$ since $H_*(\tilde{k}; \mathbb{Z})$ has these properties. This implies that $(\text{Fill}(k), K)$ is homologically 1-connected. The surjectivity of $H_1(\iota; \mathbb{Z})$ also implies $\text{im } H_1(\text{fill}(k); \mathbb{Z}) = \text{im } H_1(k; \mathbb{Z})$ and together with Remark 3.14 (i) we get property (iii). If K is not connected we can apply the construction above to all of its components. \square

Remark 3.16. (i) Observe that in the lemma above it is important to assume that the rank $\text{rk } H^1(k; \mathbb{Z})$ is measured with coefficients in \mathbb{Z} . This is due to the usage of the Hurewicz theorem and covering space theory. There is no simple analogue to the Filling Lemma with coefficients in \mathbb{Z}_2 since e.g. the double cover map $k: S^1 \rightarrow S^1$ satisfies $\text{rk } H^1(k; \mathbb{Z}_2) = 0$ but cannot be filled.

Actually we could finish the proof of Theorem 1.4 right now but we want to introduce the language of *cycle spaces* which offer a more conceptual viewpoint.

In the following part two kinds of chain complexes will appear, namely singular and the simplicial chain complexes and it should always be clear from the context which one we mean depending on whether we apply it to topological spaces or simplicial sets. Nevertheless in

order to avoid confusion we will consistently try to denote the singular chain complex by C_* and the simplicial chain complex by C_\bullet .

Let $f: M^n \rightarrow N^q$ be a smooth map between closed R -oriented manifolds, σ a smooth embedded k -simplex in N which intersects f stratum transversally. Recall Proposition 3.6 by which we can assign to every vertex v of σ an $(n-q)$ -cycle c_v in M and to any l -dimensional face τ of σ an $(n-q+l)$ -chain c_τ such that we have

$$\partial c_\tau = \sum_{i=0}^l c_{\partial_i \tau}.$$

This motivates the following

Definition 3.17 (cf. [9, Section 2.2]). Let (D_*, ∂) be a chain complex of abelian groups. The *space of $(n-q)$ -cycles in D_** is a simplicial set denoted by $cl^{n-q}(D_*, \partial)$ the level sets of which are given by

$$(cl^{n-q}(D_*, \partial))_k := (cl^{n-q}D_*)_k := \text{Hom}(C_\bullet \Delta[k], D_{*+(n-q)}).$$

Some explanations are in order.

- (i) $\Delta[k]$ denotes the k -dimensional standard simplex in the category **sSet**.
 - (ii) $C_\bullet \Delta[k]$ denotes its normalised chain complex, i.e. the chain groups are generated only by the non-degenerate simplices of $\Delta[k]$.
 - (iii) The Hom set is meant as the set of morphisms of chain complexes of abelian groups.
- The right hand side defines a contravariant functor $\Delta \rightarrow \mathbf{Set}$ where Δ is the ordinal number category. This turns $cl^{n-q}D_*$ into a simplicial set.

The main example of a chain complex D_* to which we want to apply the construction above is the singular chain complex of the source manifold, e.g. a torus.

Remark 3.18. (i) The chain groups $C_i \Delta[k]$ are non-zero only for $0 \leq i \leq k$ and $C_k \Delta[k] \cong \mathbb{Z}$. Let $c_k \in C_k \Delta[k]$ be a generator.

- (ii) Thus a 0-simplex σ in $cl^{n-q}D_*$ corresponds to a diagram of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & C_0 \Delta[0] & \longrightarrow & 0 \\ \downarrow & & \sigma_0 \downarrow & & \downarrow \\ D_{(n-q)+1} & \longrightarrow & D_{n-q} & \longrightarrow & D_{(n-q)-1}. \end{array}$$

This diagram is uniquely determined by the image $\sigma_0 c_0 \in C_{n-2}$ and this element satisfies $\partial \sigma_0 c_0 = 0$. So 0-simplices are in bijection to $(n-q)$ -cycles of D_* .

- (iii) Any k -simplex σ of $cl^{n-q}D_*$ is precisely a diagram of the following form.

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_k \Delta[k] & \longrightarrow & \dots & \longrightarrow & C_0 \Delta[k] \longrightarrow 0 \\ \downarrow & & \sigma_k \downarrow & & & & \downarrow \sigma_0 \\ D_{(n-q)+k+1} & \longrightarrow & D_{(n-q)+k} & \longrightarrow & \dots & \longrightarrow & D_{n-q} \longrightarrow D_{(n-q)-1}. \end{array}$$

- (iv) The simplicial set $cl^{n-q}D_*$ depends covariantly on the chain complex argument D_* , turning cl^{n-q} into a covariant functor $cl^{n-q}: \mathbf{ChainCom} \rightarrow \mathbf{sSet}$.

We need some preparation in order to rigorously prove that – like we tried to motivate – Proposition 3.6 yields elements in $(cl^{n-q}C_*M)_k$.

Lemma 3.19. There are evaluation maps

$$\begin{aligned} \text{ev}_k &: (cl^{n-q}D_*)_k \rightarrow D_{(n-q)+k} \\ \sigma &\mapsto \sigma_k c_k \end{aligned}$$

and these extend and fit together such that

$$\text{ev}_\bullet : C_\bullet cl^{n-q}D_* \rightarrow D_{\bullet+(n-q)}$$

is a morphism of chain complexes. The source of ev_\bullet is the simplicial chain complex of $cl^{n-q}D_*$. Sometimes we abbreviate $\text{ev}_k\sigma$ by $\widehat{\sigma}$.

Proof. We have to check the commutativity of

$$\begin{array}{ccc} C_k cl^{n-q}D_* & \xrightarrow{\text{ev}_k} & D_{(n-q)+k} \\ \partial \downarrow & & \downarrow \partial \\ C_{k-1} cl^{n-q}D_* & \xrightarrow{\text{ev}_{k-1}} & D_{(n-q)+k-1}. \end{array}$$

and this is equivalent to

$$\text{ev}_{k-1}\partial\sigma = \partial\text{ev}_k\sigma$$

for every $\sigma \in (cl^{n-q}D_*)_k$. We have

$$\begin{aligned} \text{ev}_{k-1}\partial\sigma &= \sum_i (-1)^i \partial_i \sigma \\ &= \sum_i (-1)^i \text{ev}_{k-1} \partial_i \sigma. \end{aligned}$$

The faces $\partial_i \sigma \in (cl^{n-q}D_*)_{k-1}$ are given by the following precomposition with the inclusion of faces $d_i : \Delta[k-1] \rightarrow \Delta[k]$.

$$\begin{array}{ccccccc} & & C_{k-1}\Delta[k-1] & \longrightarrow & \dots & \longrightarrow & C_0\Delta[k-1] \\ & & \downarrow C_{k-1}d_i & & & & \downarrow \\ 0 & \longrightarrow & C_k\Delta[k] & \longrightarrow & C_{k-1}\Delta[k] & \longrightarrow & \dots \longrightarrow C_0\Delta[k] \longrightarrow 0 \\ \downarrow & & \downarrow \sigma_k & & \downarrow & & \downarrow \sigma_0 \\ D_{(n-q)+k+1} & \longrightarrow & D_{(n-q)+k} & \longrightarrow & D_{(n-q)+k-1} & \longrightarrow & \dots \longrightarrow D_{n-q} \longrightarrow C_{(n-q)-1}. \end{array} \quad (3.9)$$

From this we can continue

$$\begin{aligned} \text{ev}_{k-1}\partial\sigma &= \sum_i (-1)^i \text{ev}_{k-1} \partial_i \sigma \\ &= \sum_i (-1)^i \sigma_{k-1} \partial_i c_k = \sigma_{k-1} \partial c_k = \partial \sigma_k c_k = \partial \text{ev}_k \sigma. \quad \square \end{aligned}$$

The following lemma shows how $(k+1)$ simplices $\varphi_i \in (cl^{n-q}D_*)_k$ can be glued together to form the faces of a simplex $\sigma \in (cl^{n-q}D_*)_{k+1}$ if the obvious homological restriction in D_* vanishes.

Lemma 3.20 (Gluing Lemma). Let $\varphi_0, \dots, \varphi_{k+1} \in (cl^{n-q}D_*)_k$ such that

$$\partial_i \varphi_j = \partial_{j-1} \varphi_i, \quad 0 \leq i < j \leq k+1.$$

If there exists an element $\bar{\sigma} \in D_{(n-q)+k+1}$ with

$$\partial \bar{\sigma} = \sum_{i=0}^{k+1} (-1)^i \widehat{\varphi}_i$$

there is a unique $\sigma \in (cl^{n-q}D_*)_{k+1}$ satisfying $\widehat{\sigma} = \bar{\sigma}$ and $\partial_i \sigma = \varphi_i$.

Proof. For every $\alpha \in cl^{n-q}D_*$ diagram (3.9) implies the identity

$$\partial \widehat{\alpha} = \sum_{i=0}^l (-1)^i \widehat{\partial_i \alpha} \quad (3.10)$$

Given the element $\bar{\sigma}$ we can consider the following diagram.

$$\begin{array}{ccccccc} C_{k+1}\Delta[k+1] & \longrightarrow & C_k\Delta[k+1] & \longrightarrow & \dots & \longrightarrow & C_0\Delta[k+1] \\ \downarrow \scriptstyle C_{k+1} \mapsto \bar{\sigma} & & \downarrow \scriptstyle \partial_{i_1} \dots \partial_{i_l} C_{k+1} \mapsto \widehat{\varphi}_i & & & & \downarrow \\ D_{(n-q)+k+1} & \longrightarrow & D_{(n-q)+k} & \longrightarrow & \dots & \longrightarrow & D_{n-q} \end{array}$$

All the lower dimensional simplices are of the form $\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}$ and they shall be mapped such that

$$\sigma : \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1} \mapsto (\partial_{i_1} \dots \partial_{i_l} \varphi_i)^\wedge$$

This image is indeed invariant under applying simplicial identities to the non-unique representation $\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}$ and we get

$$\begin{aligned} & \sigma(\partial \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}) \\ &= \sigma \left(\sum_{m=0}^{k-l} (-1)^m \partial_m \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1} \right) \\ &= \sum_{m=0}^{k-l} (-1)^m (\partial_m \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1})^\wedge \\ &= \partial (\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1})^\wedge = \partial \sigma (\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}) \end{aligned}$$

where in the passage to the last line we have applied Lemma 3.19 to $\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}$. Thus σ commutes with the differentials. \square

Construction 3.21. Recall Proposition 3.6. Let $f: M^n \rightarrow N^q$ be a smooth map between closed R -oriented manifolds, \mathcal{T} an R -oriented triangulation of N such that f intersects all the simplices $\sigma \in \mathcal{T}_q$ stratum transversally. We can assign to any $\sigma \in \mathcal{T}_k$ a singular chain $c_\sigma \in C_{n-q+k}(F_\sigma; R)$ such that the following properties hold:

- (i) For $\sigma \in \mathcal{T}_0$ the chain $c_\sigma \in C_{n-q}(F_\sigma; R)$ represents the (correctly oriented) fundamental class of F_σ .
- (ii) For $1 \leq k \leq q$ and $\sigma \in \mathcal{T}_k$ we have

$$\partial c_\sigma = \sum_{i=0}^k (-1)^i c_{\partial_i \sigma}$$

as an equation in $C_{n-q+k-1}(F_\sigma; R)$ and c_σ represents the (correctly oriented) relative fundamental class in $H_{n-q+k}(F_\sigma, \partial F_\sigma; R)$.

- (iii) The sum

$$\sum_{\sigma \in \mathcal{T}_q} c_\sigma \in C_n(M; R)$$

represents the (correctly oriented) fundamental class of M .

For every $\sigma \in \mathcal{T}_0$ we can use Remark 3.18 (ii) to turn the cycles c_σ into 0-simplices $z_\sigma \in (cl^{n-q}C_*(F_\sigma; R))_0$ satisfying $\widehat{z}_\sigma = c_\sigma$.

For higher-dimensional $\sigma \in \mathcal{T}_k$ we will inductively construct elements $z_\sigma \in (cl^{n-q}C_*(F_\sigma; R))_k$ satisfying

$$\partial_i z_\sigma = z_{\partial_i \sigma} \quad \text{and} \quad \widehat{z}_\sigma = c_\sigma. \quad (3.11)$$

Assume we have constructed such simplices z_τ for all τ of dimension at most k and fix $\sigma \in \mathcal{T}_{k+1}$. For $0 \leq i < j \leq k+1$ we have

$$\partial_i z_{\partial_j \sigma} = z_{\partial_i \partial_j \sigma} = z_{\partial_{j-1} \partial_i \sigma} = \partial_{j-1} z_{\partial_i \sigma}$$

and

$$\partial c_\sigma = \sum_{i=0}^{k+1} (-1)^i c_{\partial_i \sigma} = \sum_{i=0}^{k+1} (-1)^i \widehat{z_{\partial_i \sigma}}.$$

Applying the Gluing Lemma 3.20 to $\varphi_i := z_{\partial_i \sigma}$ and $\bar{\sigma} := c_\sigma$ yields a simplex $z_\sigma \in (cl^{n-q}C_*(F_\sigma; R))_{k+1}$ with the desired properties (3.11).

The simplicial chain

$$Z(f, \mathcal{T}) := \sum_{\sigma \in \mathcal{T}_q} z_\sigma$$

can be viewed as an element in $C_q cl^{n-q}C_*(M; R)$ and it satisfies $\partial Z(f, \mathcal{T}) = 0$. Since

$$\text{ev}_\bullet: C_\bullet cl^{n-q}C_*(M; R) \rightarrow D_{\bullet+(n-q)}$$

is a morphism of chain complexes and maps $Z(f, \mathcal{T})$ to $\sum_{\sigma \in \mathcal{T}_q} c_\sigma$ we conclude that $[Z(f, \mathcal{T})] \neq 0$ in $H_q cl^{n-q}C_*(M; R)$.

Comment 3.22. (i) There is an analytic analogue to the construction above. Let $M^n \subset \mathbb{R}^N$ be a smooth closed embedded manifold, $I_k(M)$ be the topological space of integral

currents with the flat topology and $Z_k(M) \subset I_k(M)$ the subspace of cycles. In [1] Almgren proved that the homotopy groups of the latter are given by

$$\pi_i Z_k(M) \cong H_{i+k}(M).$$

A priori the homotopy groups of a space do not determine its homotopy type since it could have non-zero k -invariants but in the case of $Z_k(M)$ the topological group completion theorem implies that the k -invariants of every topological abelian monoid vanish. In particular we get

$$Z_k(M) \simeq \prod_{i=0}^{n-k} K(H_{i+k}(M), i). \quad (3.12)$$

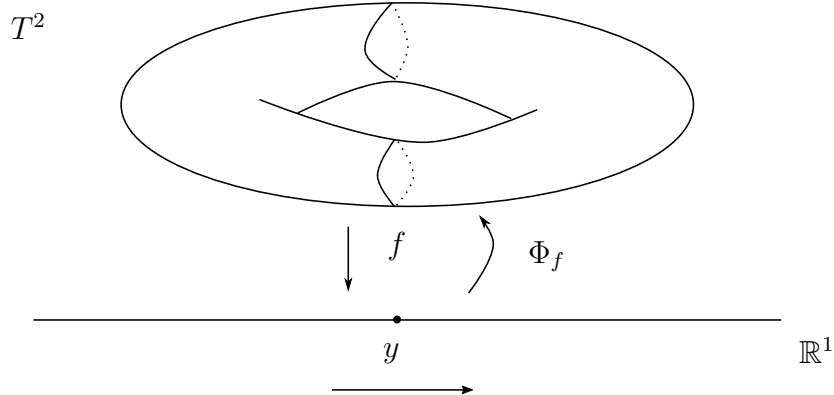
One reasonable corollary from this is $\pi_0 Z_k(M) = H_k M$. Another consequence is

$$\pi_q Z_{n-q}(M) \cong H_n(M) \cong \mathbb{Z} \quad (3.13)$$

and the generator is given as follows. Let $f: M \subset \mathbb{R}^N \rightarrow \mathbb{R}^q$ be a generic projection. For any $y \in \mathbb{R}^q$ the preimage $f^{-1}(y)$ defines an $(n - q)$ -dimensional integral cycle and the map

$$\begin{aligned} \Phi_f: \mathbb{R}^q &\rightarrow Z_{n-q}(M) \\ y &\mapsto f^{-1}(y) \end{aligned}$$

is continuous and maps everything outside of $\text{im } f$ to the zero cycle. Hence it determines an element $[\Phi_f] \in \pi_q Z_{n-q}(M)$ which is independent of f and corresponds exactly to the fundamental class under the correspondence (3.13).



It is an important observation – especially when proving waist inequalities – that every map $f: M^n \rightarrow \mathbb{R}^q$ yields a homotopically nontrivial map $\Phi_f: \mathbb{R}^q \rightarrow Z_{n-q}(M)$. There are different ways to formalise the notion of spaces of cycles. For obvious reasons we chose a definition with the flavour of algebraic topology.

- (ii) Of course one wonders what is the homotopy type of $cl^{n-q} D_*$ for a given chain complex D_* . Up to an index shift cl^{n-q} is just the Dold-Kan correspondence between chain

complexes and simplicial abelian groups and from that we get in analogy to (3.12)

$$cl^{n-q}D_* \simeq \prod_{i=0}^{\infty} K(H_{n-q+i}(D_*), i).$$

- (iii) In the construction above the cycle $Z(f, \mathcal{T})$ in $cl^{n-q}C_*(M; R)$ is called the *canonical cycle associated to f and \mathcal{T}* and $[Z(f; \mathcal{T})] \in H_q cl^{n-q}C_*(M; R)$ the *canonical homology class*. The cycle $Z(f, \mathcal{T})$ depends heavily on the map f and the triangulation \mathcal{T} whereas one can show that $[Z(f; \mathcal{T})]$ is independent of these choices. We could define the canonical homology class far easier as being represented by the q -simplex given by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_q \Delta[q] & \longrightarrow & \dots & \longrightarrow & C_0 \Delta[q] \longrightarrow 0 \\ \downarrow & & \sigma_q \downarrow & & & & \downarrow \sigma_0 \\ C_{n+1}M & \longrightarrow & C_n M & \longrightarrow & \dots & \longrightarrow & C_{n-q}M \longrightarrow D_{(n-q)-1}. \end{array}$$

where σ_q maps c_q to a fundamental cycle of M and all other c_i vanish. This cycle arises from the geometric construction above if there exists one large q -simplex containing the $\text{im } f$.

However this cycle does not incorporate the map f and the fine triangulation \mathcal{T} in such a way which will enable us to execute the proof of Proposition 3.11 which we restate for convenience.

Proposition 3.11. Let N be a closed q -manifold. There is no smooth map $f: T^n \rightarrow N$ together with a smooth triangulation \mathcal{T} of N such that the following two properties hold:

- (i) The smooth simplices of \mathcal{T} intersect f stratum transversally.
- (ii) For every $\sigma \in \mathcal{T}_k$ the inclusion $f_\sigma: F_\sigma := f^{-1}\sigma(\Delta^k) \hookrightarrow T^n$ satisfies

$$\text{rk } H^1(f_\sigma; \mathbb{Z}) < n - q. \quad (3.14)$$

In the following proof there will be a certain unpleasant mixture of coefficients between \mathbb{Z} and \mathbb{Z}_2 . After all this could not have been totally avoided since we do not want to assume the target manifold N to be orientable which introduces \mathbb{Z}_2 coefficients at some places. On the other hand, as explained in Remark 3.16 (iii), the usage of Filling Lemma 3.13 forces us to interpret some expressions, e.g. (3.14), with coefficients in \mathbb{Z} .

Proof. We proceed by contradiction and assume that such a map f and triangulation \mathcal{T} exist. Recall the simplices $z_\sigma \in (cl^{n-q}C_*(F_\sigma; \mathbb{Z}_2))_k$ and the canonical cycle

$$Z(f; \mathcal{T}) := \sum_{\sigma \in \mathcal{T}_q} z_\sigma \in C_q cl^{n-q}C_*(T^n; \mathbb{Z}_2)$$

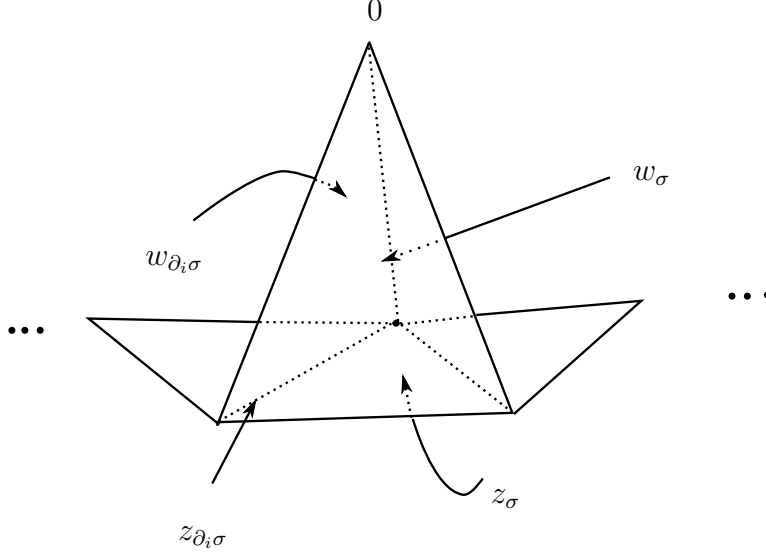
from Construction 3.21.

We will build the cone of Z inside $cl^{n-q}C_*(T^n; \mathbb{Z}_2)$. For every $\sigma \in \mathcal{T}_k$ we will construct simplices $w_\sigma \in (cl^{n-q}C_*(T^n; \mathbb{Z}_2))_{k+1}$ satisfying

$$\partial_i w_\sigma = \begin{cases} w_{\partial_i \sigma} & \text{if } 0 \leq i \leq k \\ z_\sigma & \text{if } i = k + 1. \end{cases} \quad (3.15)$$

For $\sigma \in \mathcal{T}_0$ and $i = 0$ equation (3.15) shall be interpreted as $\partial_0 w_\sigma = w_{\partial_0 \sigma} = 0$.

$$cl^{n-q}C_*(T^n; \mathbb{Z}_2)$$



If we constructed such simplices w_σ the standard cone calculation shows

$$\partial \sum_{\sigma \in \mathcal{T}_q} w_\sigma = (-1)^{q+1} Z(f; \mathcal{T})$$

contradicting Construction 3.21 where we have seen that $[Z(f; \mathcal{T})] \neq 0$ in $H_q cl^{n-q}C_*(T^n; \mathbb{Z}_2)$. So we are only left with constructing simplices w_σ satisfying equation (3.15).

Recall Notation 3.15 that every map from a topological space to T^n is denoted by the lower case letter corresponding to the upper case letter representing the space. For every $0 \leq k \leq q$ and $\sigma \in \mathcal{T}_k$ we will inductively construct triples $(L_\sigma, K_\sigma, F_\sigma)$ of topological spaces and simplices $w_\sigma \in (cl^{n-q}C_*(L_\sigma; \mathbb{Z}_2))_{k+1}$ such that the following properties hold.

- (i) (L_σ, F_σ) is a homologically 1-connected relative CW complex and we write

$$L_\sigma = F_\sigma \cup e_\sigma \tag{3.16}$$

where e_σ is an abbreviation for all the cells which we need to attach to F_σ in order to obtain L_σ .

- (ii) There are canonical inclusions as in the following diagram.

$$\begin{array}{ccc} L_{\partial_i \sigma} & \hookrightarrow & L_\sigma \\ \uparrow & \searrow & \uparrow \\ K_{\partial_i \sigma} & \hookrightarrow & K_\sigma \\ \uparrow & & \uparrow \\ F_{\partial_i \sigma} & \hookrightarrow & F_\sigma \end{array}$$

(iii) There exist extensions such that the diagram

$$\begin{array}{ccc}
 L_\sigma & & \\
 \uparrow & \searrow l_\sigma & \\
 K_\sigma & \xrightarrow{k_\sigma} & T^n \\
 \uparrow & \nearrow f_\sigma & \\
 F_\sigma & &
 \end{array}$$

commutes.

- (iv) $\text{rk } H^1(l_\sigma; \mathbb{Z}) = \text{rk } H^1(k_\sigma; \mathbb{Z}) = \text{rk } H^1(f_\sigma; \mathbb{Z}) < n - q$
- (v) We have $H_{\geq n-q}(L_\sigma; \mathbb{Z}_2) = 0$ and in particular $H_*(K_\sigma; \mathbb{Z}_2) \rightarrow H_*(L_\sigma; \mathbb{Z}_2)$ vanishes for $* \geq n - q$.
- (vi) The simplices w_σ satisfy (3.15) as a relation of simplices $cl^{n-q}C_*(L_\sigma; \mathbb{Z}_2)$. Naturally it can also be seen as a relation in $cl^{n-q}C_*(T^n; \mathbb{Z}_2)$.

In the base case $k = 0$ we can set $K_\sigma := F_\sigma$. By assumption we have $\text{rk } H^1(f_\sigma; \mathbb{Z}) < n - q$ and we can apply Filling Lemma 3.13 to it. We get a relative CW complex (L_σ, F_σ) and an extension

$$\begin{array}{ccc}
 L_\sigma & & \\
 \uparrow & \searrow l_\sigma & \\
 K_\sigma & \longrightarrow & T^n \\
 \parallel \text{id} & \nearrow f_\sigma & \\
 F_\sigma & &
 \end{array} \tag{3.17}$$

satisfying (iv). Consider the cycle $\widehat{z}_\sigma \in C_{n-q}(F_\sigma; \mathbb{Z})$ and its image under the inclusion $F_\sigma = K_\sigma \hookrightarrow L_\sigma$. Since $H_{n-q}(L_\sigma; \mathbb{Z}_2) = 0$ there exists a (suggestively denoted) chain $\widehat{w}_\sigma \in C_{n-q+1}(L_\sigma; \mathbb{Z}_2)$ such that

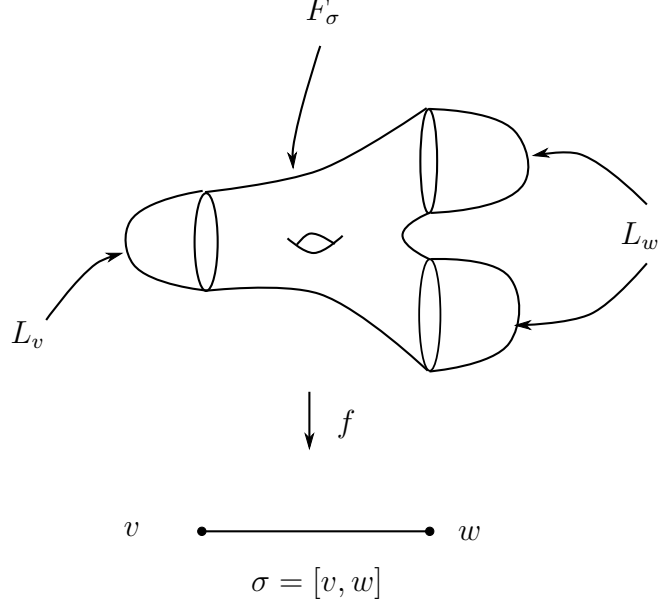
$$\partial \widehat{w}_\sigma = \widehat{z}_\sigma. \tag{3.18}$$

Using the Gluing Lemma 3.20 we get a simplex $w_\sigma \in (cl^{n-q}C_*(L_\sigma; \mathbb{Z}_2))_1$ satisfying (3.15) for $k = 0$.

Assume K_τ , L_τ and w_τ have already been constructed for all simplices τ of dimension strictly less than $k \geq 1$.

$$K_\sigma = K_\sigma^{(0)}$$

$$n - q = 1, k = 1$$



For $\sigma \in \mathcal{T}_k$ and $0 \leq i < k$ we inductively define spaces and maps $k_\sigma^{(i)}: K_\sigma^{(i)} \rightarrow T^n$ by setting $K_\sigma^{(-1)} := F_\sigma$, $k_\sigma^{(-1)} := f_\sigma$ and

$$\begin{array}{ccc} K_\sigma^{(i-1)} \cup \bigcup_{i\text{-dim. faces } \tau \text{ of } \sigma} e_\tau =: K_\sigma^{(i)} & & \\ \uparrow & \searrow k_\sigma^{(i)} & \\ K_\sigma^{(i-1)} & \xrightarrow{k_\sigma^{(i-1)}} & T^n \end{array}$$

where we used the notation introduced in equation (3.16). This is well-defined since the targets of the attaching maps of e_τ are F_τ which canonically are subspaces of $F_\sigma \subseteq K_\sigma^{(i-1)}$ for every i .

We have a homeomorphism

$$\bigvee_{i\text{-dim. faces } \tau \text{ of } \sigma} \left(L_{\partial_i \sigma} / F_{\partial_i \sigma} \right) \xrightarrow{\cong} K_\sigma^{(i)} / K_\sigma^{(i-1)}.$$

Since the pairs $(L_{\partial_i \sigma}, F_{\partial_i \sigma})$ are homologically 1-connected we conclude

$$H_* (K_\sigma^{(i)}, K_\sigma^{(i-1)}) \cong \bigoplus_{i\text{-dim. faces } \tau \text{ of } \sigma} H_* (L_{\partial_i \sigma}, F_{\partial_i \sigma}) = 0$$

for $* = 0, 1$ proving that the $(K_\sigma^{(i)}, K_\sigma^{(i-1)})$ are homologically 1-connected.

Let $K_\sigma := K_\sigma^{(k-1)}$, $k_\sigma := k_\sigma^{(k-1)}$. Since all the $(K_\sigma^{(i)}, K_\sigma^{(i-1)})$ are homologically 1-connected the same holds for (K_σ, F_σ) . In particular the inclusion $F_\sigma \hookrightarrow K_\sigma$ induces a surjective

homomorphism $H_1(F_\sigma; \mathbb{Z}) \rightarrow H_1(K_\sigma; \mathbb{Z})$. This surjectivity, Remark 3.14 (i) and the diagram

$$\begin{array}{ccc} K_\sigma & \xrightarrow{k_\sigma} & T^n \\ \uparrow & \nearrow f_\sigma & \\ F_\sigma & & \end{array}$$

show that $\text{rk } H^1(k_\sigma; \mathbb{Z}) = \text{rk } H_1(k_\sigma; \mathbb{Z}) = \text{rk } H_1(f_\sigma; \mathbb{Z}) = \text{rk } H^1(f_\sigma; \mathbb{Z})$.

In particular we have $\text{rk } H^1(k_\sigma; \mathbb{Z}) < n - q$ and we can apply Filling Lemma 3.13 to it in order to obtain the space $L_\sigma := \text{Fill}(k_\sigma)$ and the map $l_\sigma := \text{fill}(\sigma)$ satisfying (iii). The pair (L_σ, K_σ) is homologically 1-connected and with the same calculation as above we get $\text{rk } H^1(l_\sigma; \mathbb{Z}) = \text{rk } H^1(k_\sigma; \mathbb{Z})$.

Using the inclusions $L_{\partial_i \sigma} \subseteq K_\sigma$ and $F_\sigma \subseteq L_\sigma$ we can consider the chain

$$y_\sigma := \sum_{i=0}^k (-1)^i \widehat{w_{\partial_i \sigma}} + (-1)^{k+1} \widehat{z_\sigma} \in C_{n-q+k}(K_\sigma; \mathbb{Z}_2). \quad (3.19)$$

Since $\partial y_\sigma = 0$ and $H_{n-q+k}(L_\sigma; \mathbb{Z}_2) = 0$ there exists a (suggestively denoted) chain $\widehat{w_\sigma} \in C_{n-q+k+1}(L_\sigma; \mathbb{Z}_2)$ satisfying $\partial \widehat{w_\sigma} = y_\sigma$. Using the Gluing Lemma 3.20 we get a simplex $w_\sigma \in (cl^{n-q} C_*(L_\sigma; \mathbb{Z}_2))_{k+1}$ satisfying (3.15). \square

The proof above exhibits a close relationship between 1-dimensional quantities and fundamental classes and reminds very much of the statement and proof of the systolic inequality.

There is a natural generalisation of Theorem 1.4 to essential source manifolds M . We will recall this notion.

Definition 3.23 (Essentialness, cf. [7]). Let G be an abelian coefficient group and M^n be a closed connected G -oriented manifold with fundamental group $\pi_1(M) =: \pi$ and fundamental class $[M]_G \in H_n(M; G)$. Let $\Phi: M \rightarrow B\pi$ denote the classifying map of the universal cover $\widetilde{M} \rightarrow M$. The manifold M is said to be G -essential if the image

$$\begin{aligned} \Phi_*: H_n(M; G) &\rightarrow H_n(B\pi; G) = H_n(\pi; G) \\ [M]_G &\mapsto \Phi_*[M]_G \neq 0 \end{aligned}$$

does not vanish.

Theorem 3.24. Let M^m be manifold with fundamental group \mathbb{Z}^n and assume that at least one of the following properties holds:

- (i) M is \mathbb{Z}_2 -essential
- (ii) M and N are orientable and M is \mathbb{Z} -essential

Then every continuous map $f: M \rightarrow N$ admits a point $y \in N$ such that the rank of the restriction homomorphism satisfies

$$\text{rk } [H^1(M; \mathbb{Z}) \rightarrow H^1(f^{-1}y; \mathbb{Z})] \geq m - q.$$

Remark 3.25. (i) With the assumptions of the theorem above we automatically have $m \leq n$ since $H_{>n}(B\mathbb{Z}^n; G) = H_{>n}(T^n; G) = 0$. Examples of G -essential n -manifolds with fundamental group \mathbb{Z}^n ($m = n$) that are not necessarily tori are connected sums of T^n with any simply connected manifold in dimensions $n \geq 3$. If $4 \leq m < n$ we can

start with a map $\varphi: T^m \rightarrow T^n$ such that $H_m(\varphi; G)[T^m] \neq 0$ and use surgery to turn this into an essential m -manifold with fundamental group \mathbb{Z}^n . More generally, manifolds satisfying largeness conditions, such as enlargeability, are \mathbb{Z} -essential, compare [2], Theorem 3.6. in connection with Corollary 3.5.

- (ii) For orientable manifolds M^m with fundamental group \mathbb{Z}^n and classifying map $\Phi: M \rightarrow T^n$ we have the following commutative diagram.

$$\begin{array}{ccccc} H_m(M; \mathbb{Z}) & \xrightarrow{H_m(\Phi; \mathbb{Z})} & H_m(T^n; \mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^{\binom{n}{m}} \\ \downarrow & & \downarrow & & \downarrow \\ H_m(M; \mathbb{Z}_2) & \xrightarrow{H_m(\Phi; \mathbb{Z}_2)} & H_m(T^n; \mathbb{Z}_2) & \xrightarrow{\cong} & \mathbb{Z}_2^{\binom{n}{m}} \end{array}$$

The vertical arrows are change-of-coefficient homomorphisms and the leftmost vertical arrow maps $[M]_{\mathbb{Z}}$ to $[M]_{\mathbb{Z}_2}$. This diagram shows that for manifolds with free abelian fundamental group \mathbb{Z}_2 -essentialness implies \mathbb{Z} -essentialness. This explains the somehow inorganic essentialness assumption in the theorem above.

Proof of Theorem 3.24. We only discuss case (i) and indicate the necessary adaptations to the existing proof. Like we reduced Theorem 1.4 to Proposition 3.11 we proceed by contradiction and assume that N is connected and closed, there exists a smooth $f: M \rightarrow N$ and a triangulation \mathcal{T} of N such that the following two properties hold:

- (i) The smooth simplices of \mathcal{T} intersect f stratum transversally.
- (ii) For every $\sigma \in \mathcal{T}_k$ the inclusion $f_\sigma: F_\sigma := f^{-1}\sigma(\Delta^k) \hookrightarrow M$ satisfies

$$\text{rk } H^1(f_\sigma; \mathbb{Z}) < m - q.$$

Again for every $\sigma \in \mathcal{T}_k$ we consider the simplices $z_\sigma \in (cl^{m-q}C_*(F_\sigma; \mathbb{Z}_2))_k$ and the canonical cycle $Z(f; \mathcal{T}) \in C_q cl^{m-q}C_*(M; \mathbb{Z}_2)$ from Construction 3.21. Every F_σ comes with a reference map to M and naïvely we would think that we are in need of a replacement for Filling Lemma 3.13 where all the maps have target M instead of T^n . Instead consider the classifying map $\Phi: M \rightarrow T^n$. The diagram

$$\begin{array}{ccc} Z(f; \mathcal{T}) & \xrightarrow{C_q cl^{m-q}(\Phi)} & C_q cl^{m-q}(\Phi) Z(f; \mathcal{T}) \\ \downarrow \text{ev}_q & & \downarrow \text{ev}_q \\ C_\bullet cl^{m-q}C_*(M; \mathbb{Z}_2) & \longrightarrow & C_\bullet cl^{m-q}C_*(T^n; \mathbb{Z}_2) \\ \downarrow & & \downarrow \\ C_{(m-q)+*}(M; \mathbb{Z}_2) & \longrightarrow & C_{(m-q)+*}(T^n; \mathbb{Z}_2) \\ \downarrow & & \downarrow \\ \widehat{Z(f; \mathcal{T})} & \xrightarrow{C_m(\Phi)} & \Phi_* \widehat{Z(f; \mathcal{T})} \end{array}$$

commutes, the bottom left cycle represents the fundamental class $[M]_{\mathbb{Z}_2} \in H_m(M; \mathbb{Z}_2)$ and since M is \mathbb{Z}_2 -essential the bottom right cycle defines a non-zero element in $H_m(T^n; \mathbb{Z}_2)$. Therefore the top right cycle defines a non-zero element in $H_q cl^{m-q} C_*(T^n; \mathbb{Z}_2)$.

The induced map $\pi_1 \Phi$ is an isomorphism just as $H_1(\Phi; \mathbb{Z})$ by the Hurewicz theorem and $H^1(\Phi; \mathbb{Z})$ by Remark 3.14 (ii). This proves that every map $k: K \rightarrow T^n$ satisfying $\text{rk } H^1(k; \mathbb{Z}) < n - q$ also satisfies

$$\text{rk } H^1(\Phi \circ k; \mathbb{Z}) = \text{rk } H^1(k; \mathbb{Z}) < n - q.$$

Hence we can proceed as earlier and deduce a contradiction by constructing a cone of $C_q cl^{m-q}(\Phi)Z(f; \mathcal{T})$ in $cl^{m-q} C_*(T^n; \mathbb{Z}_2)$ via simplices $w_\sigma \in (cl^{n-q} C_*(T^n; \mathbb{Z}_2))_{q+1}$ satisfying

$$\partial_i w_\sigma = \begin{cases} w_{\partial_i \sigma} & \text{if } 0 \leq i \leq k \\ z_\sigma & \text{if } i = k + 1. \end{cases}$$

For $\sigma \in \mathcal{T}_0$ and $i = 0$ the equation above shall be interpreted as $\partial_0 w_\sigma = w_{\partial_0 \sigma} = 0$. □

Question 3.26. (i) Theorem 1.4, the more general Theorem 3.24 and the core input of both, Filling Lemma 3.13, give the impression that we have not proven something about tori but about the geometry of the group \mathbb{Z}^n . Are there analogues for other groups G ? Even in the case where G is abelian with torsion, this is harder because $B\mathbb{Z}_p$ has cohomology classes in arbitrary high degrees and not every cohomology class in H^*G is a product of degree 1 classes.

(ii) Michał Marcinkowski asked the entirely legitimate question whether Theorem 3.24 fails if M has fundamental group \mathbb{Z}^n but is inessential.

There is another natural generalisation of Theorem 1.4 from tori to cartesian powers of higher-dimensional spheres. Our previous proof of Filling Lemma 3.13 used covering space theory and cannot be generalised to simply connected manifolds. Instead we will use rational homotopy theory.

Theorem 1.5. Let $p \geq 3$ be odd and $n \leq p - 2$. Consider $M = (S^p)^n$ or any simply connected, closed manifold of dimension pn with the rational homotopy type $(S^p)_{\mathbb{Q}}^n$ and N^q an arbitrary orientable q -manifold. Every continuous map $f: M \rightarrow N$ admits a point $y \in N$ such that the rank of the restriction homomorphism satisfies

$$\text{rk } [H^p(M; \mathbb{Q}) \rightarrow H^p(f^{-1}y; \mathbb{Q})] \geq n - q.$$

Remark 3.27. Examples of manifolds M as above that are not $(S^p)^n$ are products of rational homology spheres of dimension p or connected sums of $(S^p)^n$ with rational homology spheres of dimension pn .

For the rest of this section the coefficient ring is always $R = \mathbb{Q}$. We assume that the reader is familiar with rational homotopy theory (cf. [5] and [6]). But before we prove the theorem above we will shortly recap the notion of *spatial realisation*:

Repetition 3.28 (Spatial realisation). There is a contravariant functor $|\cdot|: \mathbf{cgda} \rightarrow \mathbf{Top}$, called *spatial realisation*, and for every space X a continuous map

$$h_X: X \rightarrow |A_{PL}(X)|.$$

These maps are called *unit maps* and up to homotopy they are natural in X , i.e. for any continuous map $f: X \rightarrow Y$ the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h_X \downarrow & & \downarrow h_Y \\ |A_{PL}X| & \xrightarrow{|A_{PL}f|} & |A_{PL}Y| \end{array}$$

commutes up to homotopy.

The unit maps h_X are always *rational homology equivalences*, i.e. $H_*(h_X; \mathbb{Q})$ (or equivalently $H^*(h_X; \mathbb{Q})$) are isomorphisms. For any rational space $X_{\mathbb{Q}}$ and any minimal model $m: (\bigwedge V, d) \rightarrow A_{PL}X_{\mathbb{Q}}$ the maps

$$h_{X_{\mathbb{Q}}}: X_{\mathbb{Q}} \xrightarrow{\simeq} |A_{PL}X_{\mathbb{Q}}|$$

and

$$|m|: |A_{PL}X_{\mathbb{Q}}| \xrightarrow{\simeq} \left| \bigwedge V, d \right|$$

are homotopy equivalences.

Now we can start proving Theorem 1.5.

Lemma 3.29. Let $(\bigwedge[x_1, \dots, x_n], 0)$ be the minimal cgda with all generators concentrated in degree p and (A, d) an arbitrary cgda. Any morphism

$$f^{\sharp}: \left(\bigwedge[x_1, \dots, x_n], 0 \right) \rightarrow (A, d)$$

with $\text{rk } H^p(f^{\sharp}) < n - q$ satisfies $H^{\geq(n-q)p}(f^{\sharp}) = 0$.

Proof. The statement is non-vacuous only in degrees divisible by p , i.e. lp with $l \geq n - q > \text{rk } H^p(f^{\sharp})$. It suffices to prove the case $l = n - q$. Consider an arbitrary monomial of length $n - q$, without loss of generality $x_1 \cdots x_{n-q}$. Since $\text{rk } H^p(f^{\sharp}) < n - q$ there exists one factor, without loss of generality x_{n-q} , such that $[f^{\sharp}x_{n-q}]$ can be expressed as

$$[f^{\sharp}x_{n-q}] = \sum_{i < n-q} \lambda_i [f^{\sharp}x_i]$$

and hence

$$\begin{aligned} [f^{\sharp}(x_1 \cdots x_{n-q})] &= [f^{\sharp}x_1] \cdots [f^{\sharp}x_{n-q}] \\ &= [f^{\sharp}x_1] \cdots [f^{\sharp}x_{n-q-1}] \cdot \sum_{i < n-q} \lambda_i [f^{\sharp}x_i] = 0. \quad \square \end{aligned}$$

Lemma 3.30. Let $n \leq p - 2$. For any $0 \leq a < n$ the linear diophantine equation

$$\lambda(p - 1) + \mu p = np - a \tag{3.20}$$

has exactly one solution $(\lambda, \mu) \in \mathbb{Z}_{\geq 0}^2$ given by $(\lambda, \mu) = (a, n - a)$.

Proof. The integer solutions of (3.20) are parametrised by

$$\{(\lambda, \mu) = (a + kp, (n - a) - k(p - 1)) | k \in \mathbb{Z}\}.$$

Then the additional requirement $\lambda, \mu \geq 0$ translates into

$$-\frac{a}{p} \leq k \leq \frac{n - a}{p - 1}. \quad (3.21)$$

Since

$$\frac{n - a}{p - 1} - \left(-\frac{a}{p}\right) \leq \frac{n - a}{p - 1} + \frac{a}{p - 1} = \frac{n}{p - 1} < 1$$

inequality (3.21) has at most one solution. It is easy to check that $(a, n - a)$ satisfies all desired properties. \square

The lemma above will enable us to prove the following rational version of Filling Lemma 3.13.

Lemma 3.31 (Rational Filling Lemma). Let $p \geq 3$ be odd, $n \leq p - 2$, $q < n$ and $k: K \rightarrow (S^p)_{\mathbb{Q}}^n$ a continuous map with $H^p(k; \mathbb{Q}) < n - q$. There exists a relative CW complex $(\text{Fill}(k), K)$ and an extension $\text{fill}(k): \text{Fill}(k) \rightarrow (S^p)_{\mathbb{Q}}^n$ such that the diagram

$$\begin{array}{ccc} & \text{Fill}(k) & \\ \iota \uparrow & \searrow \text{fill}(k) & \\ K & \xrightarrow{k} & (S^p)_{\mathbb{Q}}^n \end{array}$$

commutes and the following properties hold.

- (i) $H_{\geq np - q}(\iota; \mathbb{Q}) = 0$
- (ii) $H^p(\text{Fill}(k), K; \mathbb{Q}) = 0$
- (iii) $\text{rk } H^p(\text{fill}(k); \mathbb{Q}) = \text{rk } H^p(k; \mathbb{Q}) < n - q$

Proof. The proof strategy is to solve the problem on the algebraic level of cgdas and then use spatial realisation to obtain the desired spaces and maps. Since p is odd we have a minimal model

$$\left(\bigwedge[x_1, \dots, x_n], 0\right) \rightarrow A_{PL}(S^p)_{\mathbb{Q}}^n$$

with generators x_i concentrated in degree p . Consider k^\sharp given by the following diagram.

$$\begin{array}{ccc} A_{PL}K & \xleftarrow{A_{PL}k} & A_{PL}(S^p)_{\mathbb{Q}}^n \\ & \nwarrow k^\sharp & \uparrow \\ & & (\bigwedge[x_1, \dots, x_n], 0) \end{array} \quad (3.22)$$

Morphisms between cdgas are denoted with a lower case letter endowed with the superindex \sharp . This notation shall hint at which continuous map we will get after spatial realisation. The

map k^\sharp can be factored as follows.

$$\begin{array}{ccc}
 & A_{PL}K & \\
 & \uparrow \iota^\sharp & \nwarrow k^\sharp \\
 (\bigwedge [H^{p-1}K \oplus \text{im } H^p(k^\sharp)], 0) & \xleftarrow{g^\sharp} & (\bigwedge [x_1, \dots, x_n], 0)
 \end{array} \tag{3.23}$$

The morphism g^\sharp is the obvious one. The map ι^\sharp can be defined by *choosing representing cocycles*, i.e. choose $y_i \in A_{PL}^{p-1}K$ such that $[y_i]$ constitutes a basis of $H^{p-1}(A_{PL}K)$ and define

$$\begin{aligned}
 \iota^\sharp: \left(\bigwedge [H^{p-1}K \oplus \text{im } H^p(k^\sharp)], 0 \right) &\rightarrow A_{PL}K \\
 [y_i] &\mapsto y_i \\
 H^p(k^\sharp)[x_i] &\mapsto k^\sharp x_i.
 \end{aligned}$$

With this definition $H^{p-1}\iota^\sharp$ is surjective and $H^p\iota^\sharp$ is injective both of which will in due course imply (ii) and (iii).

We are left to prove (i) which is equivalent to $H^{\geq pn-q}\iota^\sharp = 0$. For $0 \leq a \leq q < n$ consider a degree $pn - a$ element $x \in \bigwedge [H^{p-1}K \oplus \text{im } H^p(k^\sharp)]$. We will show that $H^{pn-a}\iota^\sharp[x] = 0 \in H^p A_{PL}K$. Without loss of generality x is a product of λ generators of degree $(p-1)$ and μ generators of degree p . Since $n \leq p-2$ Lemma 3.30 yields $(\lambda, \mu) = (a, n-a)$. Thus x contains at least $n-q$ generators of degree p , i.e.

$$x = yz_1 \cdots z_{n-q}$$

and the z_i can be written as $z_i = g^\sharp w_i$ for some $w_i \in [x_1, \dots, x_n]$. We conclude

$$\begin{aligned}
 H^{pn-a}\iota^\sharp[x] &= [\iota^\sharp x] = [\iota^\sharp(yz_1 \cdots z_{n-q})] = [(\iota^\sharp y)(\iota^\sharp z_1) \cdots (\iota^\sharp z_{n-q})] \\
 &= [(\iota^\sharp y)(\iota^\sharp g^\sharp w_1) \cdots (\iota^\sharp g^\sharp w_{n-q})] = [\iota^\sharp y] H^{(n-q)p} k^\sharp [w_1 \cdots w_{n-q}].
 \end{aligned}$$

Using the natural isomorphism $H^* A_{PL}(X) \cong H^*(X; \mathbb{Q})$ we get that $\text{rk } H^p(k^\sharp) < n-q$. Hence we can apply the Lemma 3.29 to conclude $H^{\geq (n-q)p}(k^\sharp) = 0$ proving $H^{pn-a}\iota^\sharp[x] = 0$.

Let

$$\left(\bigwedge W, 0 \right) := \left(\bigwedge [H^{p-1}K \oplus \text{im } H^p(k^\sharp)], 0 \right).$$

After spatial realisation of diagrams (3.22) and (3.23) and introducing the unit maps from Repetition 3.28 we get the following diagram.

$$\begin{array}{ccc}
 K & \xrightarrow{k} & (S^p)_{\mathbb{Q}}^n \\
 h_K \downarrow & & h_{(S^p)_{\mathbb{Q}}^n} \downarrow \simeq \\
 |A_{PL}K| & \longrightarrow & |A_{PL}(S^p)_{\mathbb{Q}}^n| \\
 |\iota^\sharp| \downarrow & & \downarrow \simeq \\
 |\bigwedge W, 0| & \xrightarrow{|g^\sharp|} & |\bigwedge [x_1, \dots, x_n], 0|
 \end{array}$$

In this diagram the lower square commutes strictly but the upper one only up to homotopy. The map h_K is a rational cohomology equivalence, in particular we still have that $H^{p-1}(|\iota^\sharp| \circ$

h_K) is surjective and $H^p(|\iota^\sharp| \circ h_K)$ is injective. The same theorem states that the right hand side vertical arrows are homotopy equivalences. After choosing homotopy inverses we get the triangle

$$\begin{array}{ccc} K & \xrightarrow{k} & (S^p)_{\mathbb{Q}}^n \\ |\iota^\sharp| \circ h_K \downarrow & \nearrow \widehat{g} & \\ |\wedge W, 0| & & \end{array}$$

which commutes up to homotopy. Choose such a homotopy $H: \widehat{g} \circ (|\iota^\sharp| \circ h_K) \simeq k$ and consider the mapping cylinder of $|\iota^\sharp| \circ h_K$.

$$\begin{array}{ccc} & & |\wedge W, 0| \\ & \nearrow & \\ M(|\iota^\sharp| \circ h_K) & & \\ & \searrow \bar{g} & \\ K & \xrightarrow{k} & (S^p)_{\mathbb{Q}}^n \end{array}$$

Using the homotopy H we get a map \bar{g} such that the diagram

$$\begin{array}{ccc} M(|\iota^\sharp| \circ h_K) & & \\ \uparrow & \searrow \bar{g} & \\ K & \xrightarrow{k} & (S^p)_{\mathbb{Q}}^n \end{array}$$

commutes strictly. Choose a relative CW approximation $(\text{Fill}(k), K) \rightarrow (M(|\iota^\sharp| \circ h_K), K)$, i.e. a relative CW complex $(\text{Fill}(k), K)$ together with a map $\text{Fill}(k) \rightarrow M(|\iota^\sharp| \circ h_K)$ which is a homotopy equivalence and restricts to the identity on K . Define ι and $\text{fill}(k)$ as in the following diagram.

$$\begin{array}{ccccc} & & \text{fill}(k) & & \\ & \nearrow & & \searrow & \\ \text{Fill}(k) & \xrightarrow{\simeq} & M(|\iota^\sharp| \circ h_K) & \longrightarrow & (S^p)_{\mathbb{Q}}^n \\ & \nwarrow \iota & \uparrow & \nearrow k & \\ & & K & & \end{array}$$

The induced map $H^{p-1}(\iota; \mathbb{Q})$ is surjective and $H^p(\iota; \mathbb{Q})$ is injective since $|\iota^\sharp| \circ h_K$ has these properties. From this we get $H^p(\text{Fill}(k), K; \mathbb{Q}) = 0$ hence $H_p(\text{Fill}(k), K; \mathbb{Q}) = 0$. As usual we successively conclude that $H_p(\iota; \mathbb{Q})$ is surjective and $\text{rk } H^p(\text{fill}(k); \mathbb{Q}) = \text{rk } H^p(k; \mathbb{Q})$. \square

Remark 3.32. (i) In the case $p = 1$ the factorisation (3.23) reminds us of our original Filling Lemma 3.13.

- (ii) The condition $n \leq p - 2$ seems a little inorganic. But in the case $n = p - 1$ the element x could be of degree np and therefore the product of p generators of degree $(p - 1)$ and we would not have any control over the image $H^{np}l^\sharp[x]$. We do not know how to weaken this condition. This may be possible by altering the construction of the Rational Filling Lemma.
- (iii) If p is even a minimal model of $(S^p)^n$ is given by $(\bigwedge[x_1, \dots, x_n, y_1, \dots, y_n], d)$ with $dy_i = x_i^2$. However it is not clear what the image of y_i under the map g^\sharp should be such that diagram (3.23) commutes or how to alter the construction.
- (iv) It is remarkable that the Rational Filling Lemma can be proven while almost exclusively manipulating algebraic objects.

Proof of Theorem 1.5. We will only indicate how to change the existing proof scheme. Again we proceed by contradiction and assume that N connected and closed, there exists a smooth $f: M^{np} \rightarrow N^q$ and a triangulation \mathcal{T} of N such that the following two properties hold:

- (i) The smooth simplices of \mathcal{T} intersect f stratum transversally.
- (ii) For every $\sigma \in \mathcal{T}_k$ the inclusion $f_\sigma: F_\sigma := f^{-1}\sigma(\Delta^k) \hookrightarrow M$ satisfies

$$\text{rk } H^p(f_\sigma; \mathbb{Q}) < n - q.$$

Again for every $\sigma \in \mathcal{T}_k$ we consider the simplices $z_\sigma \in (cl^{np-q}C_*(F_\sigma; \mathbb{Q}))_k$ and the canonical cycle $Z(f; \mathcal{T}) \in C_q cl^{np-q}C_*(M; \mathbb{Q})$ from Construction 3.21. Let $r_M: M \rightarrow (S^p)_{\mathbb{Q}}^n$ be the rationalisation map of M . The diagram

$$\begin{array}{ccc}
Z(f; \mathcal{T}) & \xrightarrow{C_q cl^{np-q}(r_M)} & C_q cl^{np-q}(r_M)Z(f; \mathcal{T}) \\
\downarrow \text{ev}_q & & \downarrow \text{ev}_q \\
& C_\bullet cl^{np-q}C_*(M; \mathbb{Q}) \longrightarrow C_\bullet cl^{np-q}C_*((S^p)_{\mathbb{Q}}^n; \mathbb{Q}) & \\
& \downarrow \qquad \qquad \qquad \downarrow & \\
& C_{(np-q)+*}(M; \mathbb{Q}) \longrightarrow C_{(np-q)+*}((S^p)_{\mathbb{Q}}^n; \mathbb{Q}) & \\
\downarrow & & \downarrow \\
\widehat{Z(f; \mathcal{T})} & \xrightarrow{C_{np}(r_M)} & C_{np}(r_M)\widehat{Z(f; \mathcal{T})}
\end{array}$$

commutes. The bottom left cycle represents the fundamental class $[M]_{\mathbb{Q}} \in H^{np}(M; \mathbb{Q})$ and by definition r_M is a rational homology equivalence, in particular the bottom right cycle defines a non-zero element in $H_{np}((S^p)_{\mathbb{Q}}^n; \mathbb{Q})$. Therefore the top right cycle defines a non-zero element in $H_q cl^{np-q}C_*((S^p)_{\mathbb{Q}}^n; \mathbb{Q})$.

Now we can use the Rational Filling Lemma 3.31, proceed as earlier and deduce a contradiction by constructing a cone of $C_q cl^{np-q}(r_M)Z(f; \mathcal{T})$ in $cl^{np-q}C_*((S^p)_{\mathbb{Q}}^n; \mathbb{Q})$ via simplices $w_\sigma \in (cl^{np-q}C_*((S^p)_{\mathbb{Q}}^n; \mathbb{Q}))_{q+1}$ satisfying

$$\partial_i w_\sigma = \begin{cases} w_{\partial_i \sigma} & \text{if } 0 \leq i \leq k \\ z_\sigma & \text{if } i = k + 1. \end{cases}$$

For $\sigma \in \mathcal{T}_0$ and $i = 0$ the equation above shall be interpreted as $\partial_0 w_\sigma = w_{\partial_0 \sigma} = 0$. □

REFERENCES

- [1] F. J. Almgren. The homotopy groups of the integral cycle groups. *Topology*, 1(4):257–299, 1962.
- [2] M. Brunnbauer and B. Hanke. Large and small group homology. *Journal of Topology*, 3(2):463, 2010.
- [3] C. H. Dowker. Čech cohomology theory and the axioms. *Annals of Mathematics*, 51(2):278–292, 1950.
- [4] S. Eilenberg and N. E. Steenrod. *Foundations of algebraic topology*. Princeton University Press, 1952.
- [5] Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*. Springer, 2005.
- [6] Y. Félix, J. Oprea, and D. Tanré. *Algebraic models in geometry*. Oxford University Press, 2008.
- [7] M. Gromov. Filling Riemannian manifolds. *Journal of Differential Geometry*, 18(1):1–147, 1983.
- [8] M. Gromov. Singularities, expanders and topology of maps. part 1: Homology versus volume in the spaces of cycles. *Geometric And Functional Analysis*, 19(3):743–841, 2009.
- [9] M. Gromov. Singularities, expanders and topology of maps. part 2: From combinatorics to topology via algebraic isoperimetry. *Geometric And Functional Analysis*, 20(2):416–526, 2010.
- [10] L. Guth. The waist inequality in Gromov’s work. *The Abel Prize 2008–2012*, <http://math.mit.edu/~7Elguth/Exposition/waist.pdf>, 2014.
- [11] M. W. Hirsch. *Differential topology*. Springer, 1976.
- [12] J. M. Lee. *Introduction to smooth manifolds*. Graduate Texts in Mathematics 218. Springer New York, 2003.
- [13] J. R. Munkres. *Elementary differential topology*. Princeton University Press, 1967.
- [14] L. T. Nielsen. Transversality and the inverse image of a submanifold with corners. *Mathematica Scandinavica*, 49(2):211–221, 1982.